Production Planning with Increasing Byproducts: MINLP Formulations and MILP Approximations

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Joint work with
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Outline

- **Problem description**
  - Production process involving desirable and undesirable products
  - Continuous-time problem containing nonconvex functions and integer decision variables
- **Discrete-time MINLP formulations**
  - Existing "natural" approach
  - An alternative formulation that is more accurate and easier to solve
- **MILP approximations and relaxations**
- **Performance evaluation**

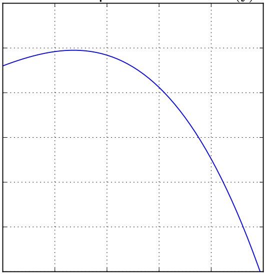
Problem Description

Production Process

- Problem contains many linked production processes, e.g., as part of a supply network, over a planning horizon $[0, T]$
  - We focus on model for a single process; in full problem, use one copy of model for each process
- Each production process creates a mixture of useful products $\mathcal{P}^+$ and undesirable byproducts $\mathcal{P}^-$
- Discrete decisions for if and when each production process starts (fixed cost)
- Continuous decisions determine the amount of mixture to produce over time
- Maximum production rate and mixture composition are functions of the cumulative total production at each process

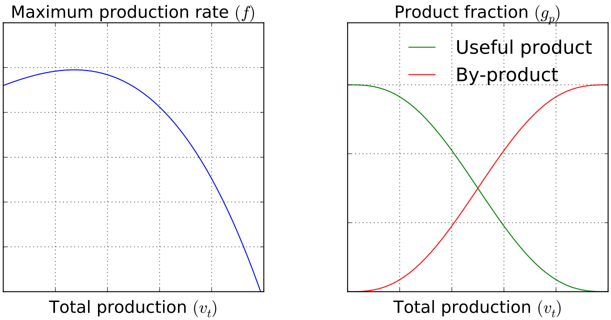
Production functions

- Production function $f(\cdot)$ is a concave function that determines the maximum production rate as a function of cumulative total production



Production functions

- Production function $f(\cdot)$ is a concave function that determines the maximum production rate as a function of cumulative total production
- Product fraction functions $g_p(\cdot)$ are monotone functions of the cumulative total production, for each $p \in \mathcal{P} = \mathcal{P}^+ \cup \mathcal{P}^-$



Continuous-time formulation

Decision variables: Production rates $x(t) \geq 0$, $t \in [0, T]$ and start-time indicator $z(t) \in \{0, 1\}$, $t \in [0, T]$

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$$y_p(t) = x(t)g_p(v(t))$$

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Production can only be positive after the process starts, $z(t) = 1$

$$v(t) \leq Mz(t)$$

Discrete-Time Formulations

Discrete-time formulations

Past models have proposed a natural discretization of this continuous-time model.

Continuous-time formulation (CNT)

$$x(t) \leq f(v(t))$$

$$v(t) = \int_0^t x(s)ds$$

$$y_p(t) = x(t)g_p(v(t))$$

$$v(t) \leq Mz(t)$$

$$z(t) : [0, T] \to \{0, 1\}, \text{increasing}$$

$\Longrightarrow$

Discrete-time formulation (F_1)

Decision variables:

- x_t Mixture production during time period $t \in \mathcal{T}$.
- v_t Cumulative production up to time period $t \in \mathcal{T}$.
- $y_{p,t}$ Product $p \in \mathcal{P}$ production during time period $t \in \mathcal{T}$.
- z_t Facility on/off decision variable.

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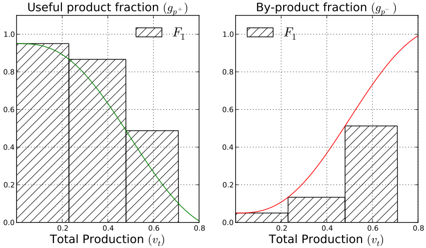
Continuous-time formulation (CNT)		Discrete-time formulation (F_1)
$x(t) \leq f(v(t))$		$x_t \leq \Delta_t f(v_{t-1})$
$v(t) = \int_0^t x(s)ds$	$\Longrightarrow$	$v_t = \sum_{s=0}^{t} x_s$
$y_p(t) = x(t)g_p(v(t))$		$y_{p,t} = x_t g_p(v_{t-1})$
$v(t) \leq Mz(t)$		$v_t \leq Mz_t$
$z(t) : [0, T] \rightarrow \{0, 1\}$, increasing		$z_t \geq z_{t-1}$

Drawbacks of the natural discrete-time formulation F_1

1. Piecewise-constant representation of product amounts is inaccurate

$$y_{p,t} = x_t g_p(v_{t-1})$$



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- Improvements are possible using midpoints, but errors will persist

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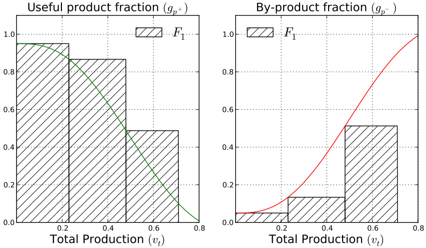

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2. Production amounts are a nonconvex function of two variables
   - Nasty to obtain global solutions or bounds

Alternate formulation

- Given *cumulative total production* v_t, we can calculate exactly how much of product $p \in \mathcal{P}$ is produced up to and including time period t:

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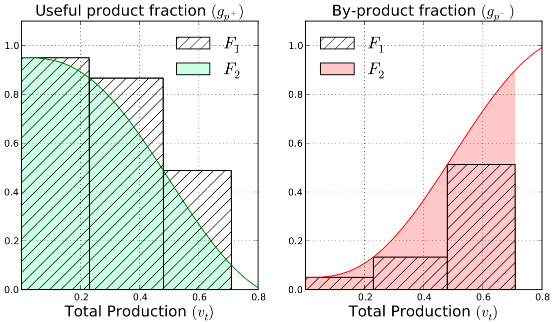
- Then, the amount of product $p \in \mathcal{P}$ produced during discrete time period t is exactly

$$y_{p,t} = h_p(v_t) - h_p(v_{t-1})$$



Alternate formulation – Another advantage

Original Formulation F_1:

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Alternative Formulation F_2:

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Computational advantages of alternative formulation

- Deals only with differences of functions of a *single* variable

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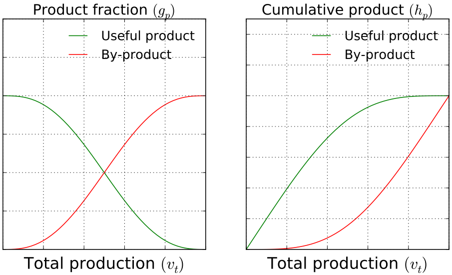
Computational advantages of alternative formulation

- Deals only with differences of functions of a *single* variable
- g_p monotone increasing $\Rightarrow h_p(v_t) = \int_0^{v_t} g_p(v)dv$ is convex!
- g_p monotone decreasing $\Rightarrow h_p(v_t) = \int_0^{v_t} g_p(v)dv$ is concave!



MIP Approximation and Relaxations

Approximations and Relaxations

Challenge

Even though h_p are convex (or concave), the constraints

$$y_{p,t} = h_p(v_t) - h_p(v_{t-1})$$

are still nonconvex

- Obtaining good bounds using general-purpose techniques can still be difficult

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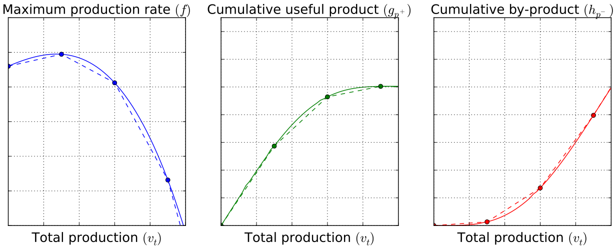
Our approach

Use piecewise linear modeling to obtain MIP approximations and relaxations

- Reason for hope: Only need to approximate "mild" univariate functions

Piecewise Linear Approximation (PLA)

Approximate all the nonlinear production functions with piecewise linearizations



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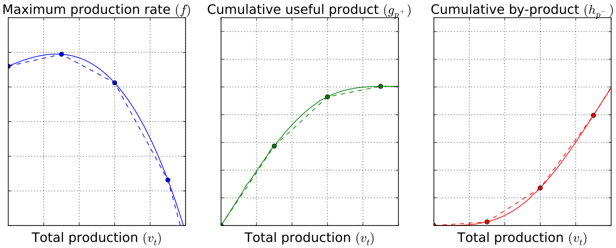
- Pros
  - ‘Close’ to a feasible solution of the MINLP formulation
  - Fixing integer decisions, then solving continuous NLP may yield a good solution



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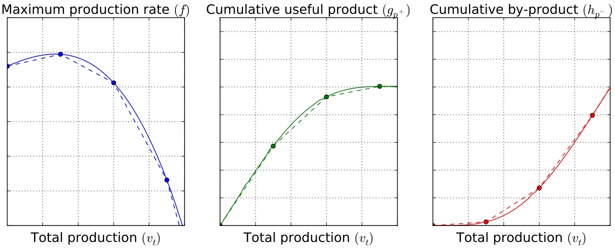
- Cons
  - Introduces additional SOS2 variables to branch on
  - NOT a relaxation of the original formulation



Piecewise Linear Approximation (PLA)

Formulation F_2

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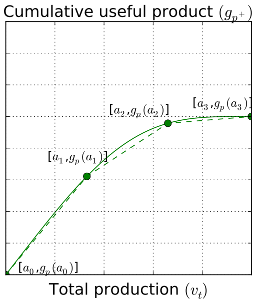

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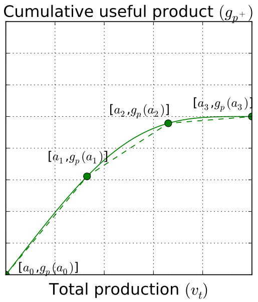

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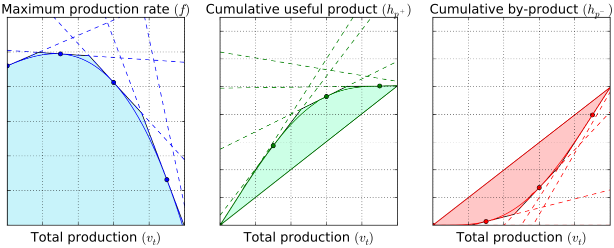
$$y_{p,t} = w_{p,t} - w_{p,t-1}$$
$$w_{p,t} = \sum_{o \in \mathcal{O}} \mathrm{H}_{p,o} \ \lambda_{t,o}$$
$$1 = \sum_{o \in \mathcal{O}} \lambda_{t,o}$$
$$\{\lambda_{t,o} | o \in \mathcal{O}\} \text{ is SOS2}$$

Secant Relaxation (1-SEC)

Relax all the nonlinear production functions using inner and outer approximations.

- Pros
  - Relaxation of the original formulation.
  - Does NOT introduce additional SOS2 variables.



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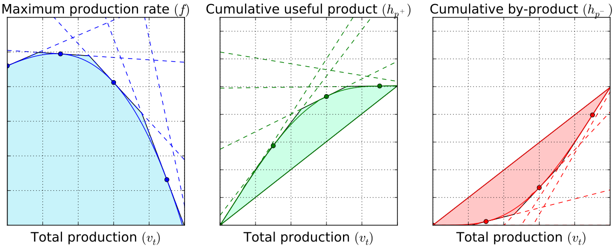

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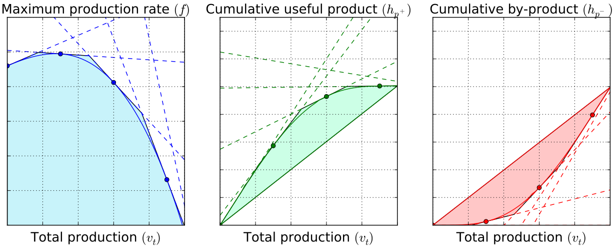

Formulation is similar to PLA, except data is different and SOS2 restriction is omitted

Multiple Secant Relaxation (k-SEC)

Relax all the nonlinear production functions using inner and outer approximations but use multiple secants instead of a just a single one.

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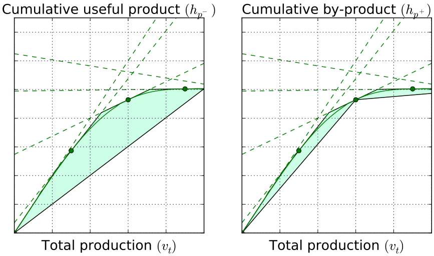

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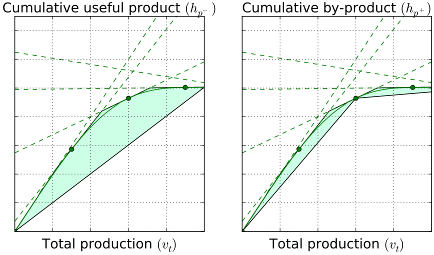


Performance Evaluation

Experiments

Goals

- Compare formulation accuracy between F_1 and F_2
- Compare solution time between F_1 and F_2

Sample Application

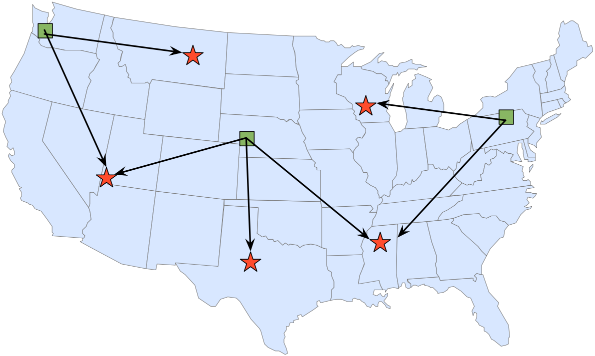
Multiple period production and distribution problem with fixed costs for opening production facilities that supply products to customers

Solvers

- To solve F_1: BARON 9.3.1
- To solve MIP approximations and linearizations of F_2: Gurobi 4.5.1
- To solve NLPs with integer variables fixed: KNITRO 3.14

Test Problem

- Multiple period production and distribution problem with production facilities $\mathcal{I}$ manufacturing products $\mathcal{P}^+$ for customers $\mathcal{J}$



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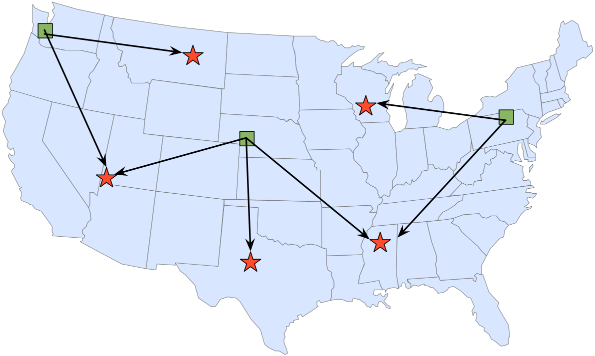

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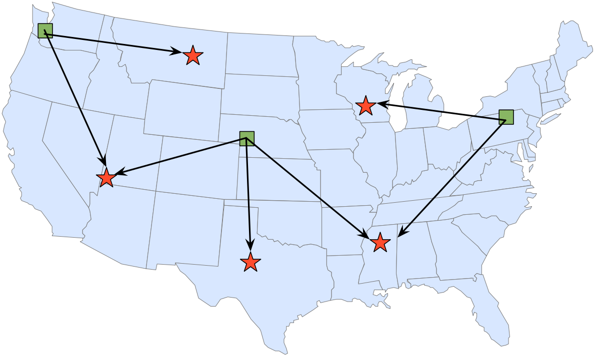

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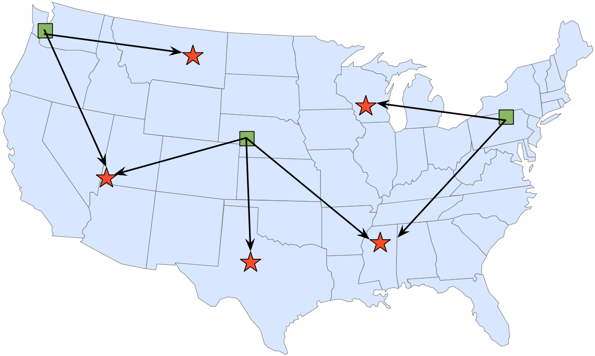
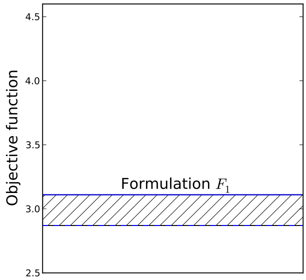
- Fixed costs for opening facilities and variable operating and transportation costs



Accuracy

Assessing impact of inaccuracy formulation F_1



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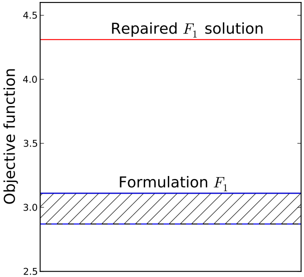

- Solve formulation F_1 (approximately)
- Repair the solution
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  - Re-solve the transportation problem

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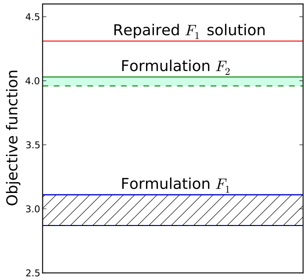

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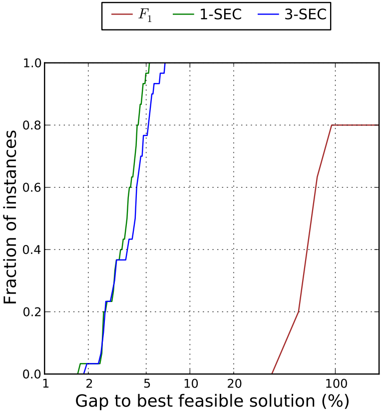
- Compare to F_2 solution
  - F_1 yields solutions 7 - 50% more costly!

Computational Efficiency

Quality of Lower Bounds



After one hour time limit

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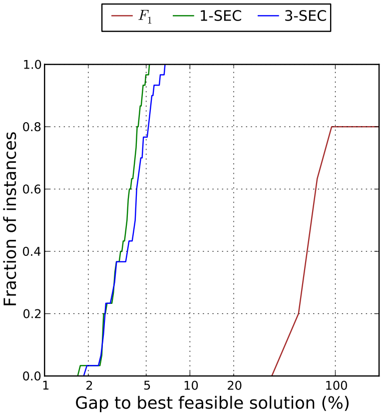
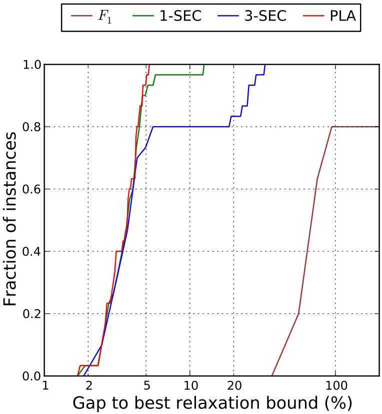


Quality of Solutions



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Conclusions

- **Problem Description**
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  - Ratio of byproducts to total production increases monotonically

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  - Defined a nonconvex production process involving desirable and undesirable products
  - Ratio of byproducts to total production increases monotonically
- **Methods**
  - Introduced a new discrete-tiem formulation (F_2) based on the cumulative product production amounts that is more accurate and computationally attractive than "natural" approach
  - Devised scalable MIP approximations and relaxations (PLA, 1-SEC, k-SEC)

Strengthening the MIP Formulations

Key Idea

- Production functions are positive only if the facility is open
- Applies to the 1-SEC, PLA and k-SEC models

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Stronger Formulation...

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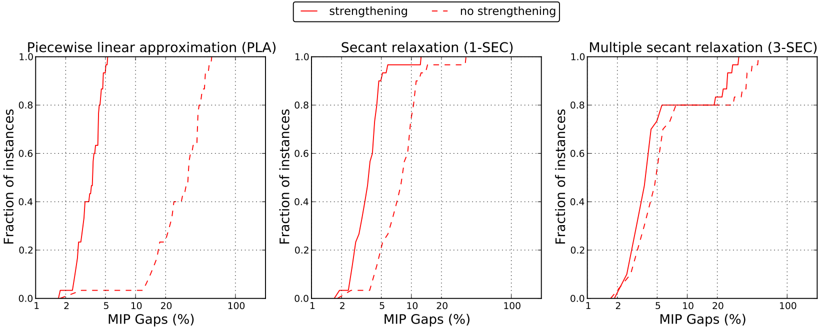
$$z_t = \sum_{o \in \mathcal{O}} \lambda_{t,o}$$

Effect of MIP Formulation Strengthening



Gaps obtained after one hour time limit when solving the MIP Formulations