Cutting Planes for the Multi-Stage Stochastic Unit Commitment Problem

Ruiwei Jiang[†], Yongpei Guan[†][*], and Jean-Paul Watson[‡]

†Department of Industrial and Systems Engineering
University of Florida, Gainesville, FL 32611

‡Discrete Math and Complex Systems Department
Sandia National Laboratories, Albuquerque, NM 87185

Abstract

As renewable energy penetration rates continue to increase in power systems worldwide, new challenges arise for system operators in both regulated and deregulated electricity markets to solve the security constrained unit commitment problem with intermittent generation (due to renewables) and uncertain load, in order to ensure system reliability and maintain cost effectiveness. In this paper, we study a security constrained multi-stage stochastic unit commitment (MSUC) model, which we use to enhance the reliability unit commitment process for day-ahead and look-ahead power system operations. In our approach, we first develop a scenario tree-based deterministic equivalent formulation for the problem, which leads to a large-scale mixed-integer linear program (MILP). By exploring substructures of the MSUC formulation, we develop several families of strong valid inequalities. In particular, we obtain (i) a convex hull representation of the minimum up/down time polytope under the stochastic scenario tree setting, (ii) strong valid inequalities to strengthen the ramping constraints by exploring the sequence independent lifting procedure, and (iii) strong valid inequalities for the general economic dispatch polytope by exploring sequence independent and subadditive approximation properties. Finally, a branch-and-cut algorithm is developed to employ these valid inequalities as cutting planes to solve the MSUC problem. Our computational results verify the effectiveness of the proposed approach.

Key words: security constrained unit commitment; stochastic programming; cutting planes; sequence independent lifting

[*]Corresponding author: guan@ise.ufl.edu

1 Introduction

Unit commitment (UC) is a fundamental optimization problem in power system operations (see, e.g., [25]), in which a system operator determines the on/off status and the power output levels for each generator at each time unit over a given operational time horizon, such that loads are satisfied with minimum total cost (see, e.g., [10] and [29]). Besides satisfying the loads in each time unit at each bus, other physical restrictions are considered, including the upper/lower output limit, the minimum up/down time, and the up/down ramp-rate limit of each generator, as well as the flow limit of each transmission line. Recently, due to the increased penetration rates of intermittent renewable energy and the introduction of demand response programs, volatilities and uncertainties on both the supply and demand sides of a power system have been increased (see, e.g., [29] and [7]), which brings extra challenges for power system operators. To prevent load shedding and black-outs, most ISOs/RTOs (e.g., Midwest-ISO and ERCOT) forecast the real time net load (e.g., the actual load minus the intermittent renewable generation if renewable generation curtailment is not allowed) and then perform day-ahead and look-ahead reliability unit commitment (RUC)[1] runs. The day-ahead RUC runs (see, e.g., [32]) are executed after the day-ahead financial market closes, and ensures that sufficient generation capacity is available to accommodate (projected) real time net load fluctuations. Accordingly, the look-ahead RUC runs (see, e.g., [19]) are executed hours ahead to ensure that sufficient numbers of fast-start generators (i.e., peaker units) are available to accommodate (projected) real time net load fluctuations. Most ISOs/RTOs utilize these two RUC runs to ensure power system reliability. A traditional approach for these RUC runs is to impose reserve requirements to achieve reliability goals, as opposed to explicitly addressing the various sources of uncertainty. That is, in current practice, RUC uses reserve margins instead of treating stochastics explicitly. Similar approaches have also been utilized for vertically integrated utilities (e.g., FPL and APS).

Recently, stochastic optimization approaches have been utilized to enhance RUC runs, while maintaining cost effectiveness. The two-stage stochastic optimization formalism accurately reflects the day-ahead RUC process, in which the first stage provides the unit commitment decisions and the second stage provides the generator dispatch levels. The objective is to minimize the total

[1]It is named Reliability Unit Commitment Process at ERCOT and Reliability Assessment Commitment Process at Midwest-ISO.

expected cost and the uncertain problem parameter (e.g., uncertain net load) is approximated by a finite number of scenarios. Significant research progress has been made recently on developing innovative security constrained two-stage stochastic UC models accommodating renewable energy generation. Instances include a security-constrained UC model taking into account intermittent wind power [29], a stochastic UC model considering various wind power forecasts with different levels of reserve requirements [28], a stochastic programming model combining slow-start generator commitment in day-ahead and fast-start generator commitment in real time operations [21], and a two-stage chance constrained stochastic programming model ensuring high utilization of wind power output [30].

Compared to two-stage stochastic UC models, multi-stage stochastic UC models can help smooth the boundary conditions (e.g., the unit on/off status mismatch at the beginning and end of an operational time horizon between runs for two consecutive operating days) in the case of day-ahead RUC runs, and allow incorporation of multi-stage forecasting information with varying accuracy, e.g., from one day to several hours ahead (see, e.g., [2]) in the case of look-ahead RUC runs. The latter provides more efficient decisions because the forecast of renewable generation output becomes more accurate as the time horizon shrinks. In addition, multi-stage stochastic UC approaches, e.g., the scenario tree-based approach, allow us to explicitly model the dependency between time periods, reflecting current management practices of renewable power generation (e.g., wind and its associated ramping restrictions). Multi-stage stochastic UC (MSUC) formulations were originally proposed in the 1990s, addressing load uncertainty while ignoring transmission constraints. For instance, in [26], an augmented Lagrangian decomposition framework to solve MSUC is introduced. More recently, in [34], the security constrained MSUC formulations have been studied. In addition, in [5], a multi-stage stochastic UC model is proposed for a power generation company that takes part in an electricity spot market. The performance of four solution approaches is compared: a direct commercial software product, a standard Lagrangean relaxation algorithm, and two original variants of Benders decomposition for multi-stage stochastic integer programs. In this paper, we consider a similar multi-stage stochastic UC formulation, i.e., the MSUC with transmission constraints, with the purpose of enhancing both day-ahead and look-ahead RUC runs.

Scenario tree-based MSUC formulations usually lead to large-scale deterministic equivalent mixed-integer linear programs (MILPs), which are hard to solve in general. Cutting plane approaches, which develop strong valid inequalities as cutting planes to tighten the linear programming relaxation of the original mixed-integer linear program, can be leveraged to efficiently solve mixed-integer linear programs [20]. In the case of stochastic mixed-integer linear program (SMILP), significant research progress has been made to solve two-stage variant. In [15], two-stage SMILPs with simple integer recourse have been proposed, using a solution approach based on the construction of the convex hull of the second-stage value function. In [16], an integer L-shaped method in which optimality cuts approximate the non-convex second-stage value function for a given binary first-stage solution is proposed. A slightly more general stochastic mixed-integer linear program is studied in [3]. In [24], a decomposition-based algorithm is developed for the solution of the two-stage SMILPs emphasizing decomposition among the integer variables that appear in the first and second stages. In [1], a branch-and-bound algorithm for two-stage stochastic integer programs with mixed-integer first-stage variables and pure integer second-stage variables is proposed.

There has been limited research on developing solution approaches for general multi-stage SMILPs. In [17], a heuristic in which the progressive hedging algorithm is combined with a tabu search is proposed to solve multi-stage SMILPs. In [4], a Lagrangian relaxation approach is proposed and implemented within a branch-and-bound algorithm for multi-stage SMILPs. In [18], a branch-and-price methodology is proposed to solve multi-stage SMILPs. In [11], a general framework is described to derive efficient cutting planes utilizing the scenario tree structure for multi-stage SMILPs, with specific applications to multi-stage stochastic lot-sizing problems [12]. In [13], the value of the multi-stage stochastic capacity planning is investigated.

In this paper, we propose novel cutting planes to solve the MSUC problems more efficiently. Specifically, we introduce several classes of strong valid inequalities to strengthen the linear programming relaxations of the MSUC formulations, which we obtain by studying the substructures of the constraints. Then we incorporate the resulting inequalities as cutting planes into a branch-and-cut framework to solve the MSUC problems efficiently. Finally, the effectiveness of our approach is analyzed on a number of test instances. To the best of our knowledge, this research represents one of the first studies developing efficient cutting planes to solve the MSUC. The proposed cutting planes can additional be integrated with other decomposition methods. We summarize our contributions of this paper as follows:

(i) We introduce a convex hull representation of the minimum up/down time polytope under the multi-stage stochastic scenario tree setting. The number of constraints in the representation is a polynomial function of the input size of the problem.

(ii) We derive strong valid inequalities to strengthen the generation ramping constraints, by exploring sequence independent lifting properties and deriving strong valid lifted inequalities. The lifted inequalities are in closed-form and the numbers for these inequalities are a polynomial function of the input size of the problem.

(iii) We derive strong cover inequalities for the economic dispatch polytope by using the lifting procedure and by deriving sequence independent and subadditive approximation lifting properties. Although the separations for these inequalities are NP-hard in general, we provide efficient heuristic separation algorithms, which are shown effective based on the computational results described in Section 6.

The remainder of this paper is organized as follows. In Section 2, we introduce the notation and mathematical formulation of the MSUC problem. For the given formulation, we develop strong valid inequalities for its substructures, in Sections 3, 4, and 5. Using the resulting inequalities as cutting planes, we develop a branch-and-cut algorithm to solve the MSUC problem, and report the computational results under various parameter settings in Section 6. Finally, we provide concluding remarks in Section 7.

2 Notation and Formulation

We assume the net loads of the MSUC problem are uncertain and follow a discrete-time stochastic process with finite support. To illustrate the multi-stage stochastic scenario tree approach, we use a scenario tree $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ with T levels (stages) to describe the possible realizations of the uncertain problem parameters, as shown in Figure 1. Each node $n \in \mathcal{V}$ at stage t of the tree specifies the state of the system that can be distinguished by information available up to and including stage t. Accordingly, for each node $n \in \mathcal{V}$, we let $t(n)$ denote the corresponding time period, $\mathcal{P}(n)$ denote the nodes along the path from the root node to node n, and p_n denote the

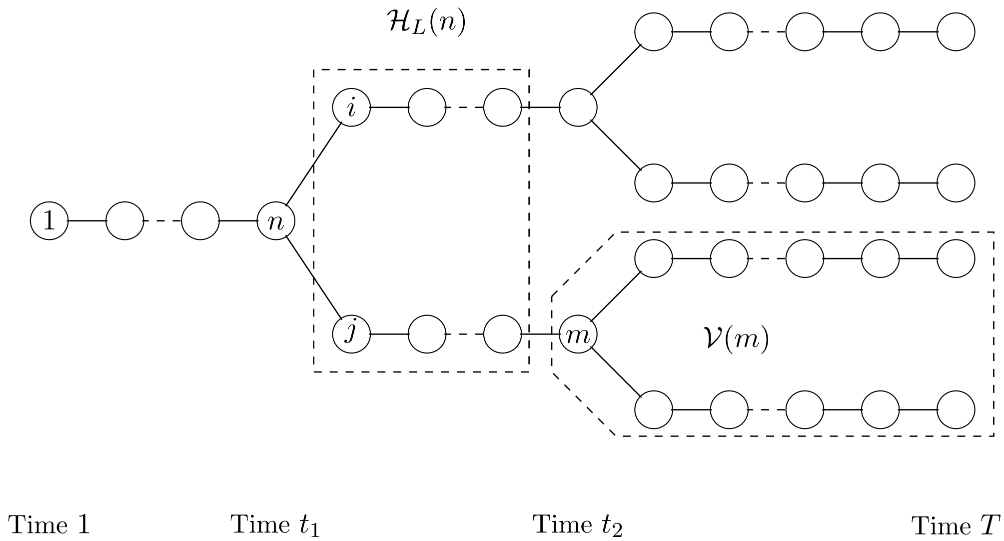

Figure 1: A multi-stage stochastic scenario tree

(absolute) probability associated with the state represented by node n. With the exception of the root node, each node n in the scenario tree has a unique parent n^-. Accordingly, all immediate children of node n are collected into set $\mathcal{C}(n)$ and all descendants are collected into set $\mathcal{V}(n)$. Finally, we let $\mathcal{H}_L(n) = \{m \in \mathcal{V}(n) : t(n) + 1 \leq t(m) \leq t(n) + L - 1\}$, where L represents the minimum up or minimum down time. To formulate the MSUC problem, we let $\mathcal{B} = \{1, \ldots, B\}$ represent the set of buses and $\mathbb{A} \subseteq \mathcal{B} \times \mathcal{B}$ represent the set of transmission lines linking two buses. We also let $\mathcal{I} = \{1, \ldots, I\}$ represent the set of generators and accordingly let $\mathcal{I}_b \subseteq \mathcal{I}$ represent the set of generators at bus b. For each transmission line $(j, h) \in \mathbb{A}$, we let C_{jh} represent the capacity of the transmission line (j, h), and let K^b_{jh} represent the line flow distribution factor for the flow on the transmission line (j, h) contributed by the net injection at bus b (see, e.g., [31] for details concerning the calculation of K^b_{jh}). For each generator $i \in \mathcal{I}$, we let $\bar{L}_i$ ($\underline{L}_i$) denote its minimum up (down) time, $\bar{Q}_i$ ($\underline{Q}_i$) denote its upper (lower) generation limit if the unit is on, V^i ($\bar{V}^i$) denote its ramp-up (start-up) rate limits, B^i ($\bar{B}^i$) denote its ramp-down (shut-down) ramp-rate limits, $\bar{S}^i$ ($\underline{S}^i$) denote its start-up (shut-down) cost, and $f^i(\cdot)$ denote the fuel cost function. Finally, we let D^b_n denote the net load of bus b at node n.

Before we describe the mathematical formulation of the MSUC problem, we introduce decision variables (y^i_n, u^i_n, v^i_n) for each node n and each generator i to represent the unit commitment decisions (e.g., $y_n^i = 1$ if the generator is on and $y_n^i = 0$ otherwise; $u_n^i = 1$ if the generator is started up and $u_n^i = 0$ otherwise; $v_n^i = 1$ if the generator is shut down and $v_n^i = 0$ otherwise), and the decision variables x_n^i to represent the generation power output level. Accordingly, the MSUC formulation is given as follows:

$$
\begin{aligned}
\min_{y,u,v,x} \quad & \sum_{n \in \mathcal{V}} p_n \Bigg(\sum_{i=1}^{I} \big(\bar{S}^i u_n^i + \underline{S}^i v_n^i + f^i(x_n^i) \big) \Bigg) && \text{(1a)}\\
\text{s.t.} \quad & y_n^i - y_{n^-}^i \le y_k^i, \quad \forall n \in \mathcal{V} \setminus \{1\}, \forall i \in \mathcal{I}, \forall k \in \mathcal{H}_{\bar{L}_i}(n), && \text{(1b)}\\
\text{(MSUC)} \quad & y_{n^-}^i - y_n^i \le 1 - y_k^i, \quad \forall n \in \mathcal{V} \setminus \{1\}, \forall i \in \mathcal{I}, \forall k \in \mathcal{H}_{\underline{L}_i}(n), && \text{(1c)}\\
& y_n^i - y_{n^-}^i \le u_n^i, \quad \forall n \in \mathcal{V} \setminus \{1\}, \forall i \in \mathcal{I}, && \text{(1d)}\\
& v_n^i = y_{n^-}^i - y_n^i + u_n^i, \quad \forall n \in \mathcal{V} \setminus \{1\}, \forall i \in \mathcal{I}, && \text{(1e)}\\
& \underline{Q}_i y_n^i \le x_n^i \le \bar{Q}_i y_n^i, \quad \forall n \in \mathcal{V}, \forall i \in \mathcal{I}, && \text{(1f)}\\
& \sum_{i=1}^{I} x_n^i = \sum_{b=1}^{B} D_n^b, \quad \forall n \in \mathcal{V}, && \text{(1g)}\\
& x_n^i - x_{n^-}^i \le (2 - y_n^i - y_{n^-}^i)\bar{V}^i + (1 + y_{n^-}^i - y_n^i)V^i, \quad \forall n \in \mathcal{V} \setminus \{1\}, \forall i \in \mathcal{I}, && \text{(1h)}\\
& x_{n^-}^i - x_n^i \le (2 - y_n^i - y_{n^-}^i)\bar{B}^i + (1 - y_{n^-}^i + y_n^i)B^i, \quad \forall n \in \mathcal{V} \setminus \{1\}, \forall i \in \mathcal{I}, && \text{(1i)}\\
& -C_{jh} \le \sum_{b=1}^{B} K_{jh}^b \Big(\sum_{i \in \mathcal{I}_b} x_n^i - D_n^b \Big) \le C_{jh}, \quad \forall n \in \mathcal{V}, \forall (j,h) \in \mathbb{A}, && \text{(1j)}\\
& y_n^i, u_n^i, v_n^i \in \{0,1\}, \quad \forall n \in \mathcal{V}, \forall i \in \mathcal{I}. && \text{(1k)}
\end{aligned}
$$

In the above formulation, the objective is to minimize the expected total cost, including start-up, shut-down, and fuel costs. Constraints (1b) and (1c) are minimum up/down time constraints. For instance, if generator i is started up at node n, then it should be kept on at all nodes in $\mathcal{H}_{\bar{L}_i}(n)$. Similarly, if generator i is shut down at node n, then it should be kept off at all nodes in $\mathcal{H}_{\underline{L}_i}(n)$. Constraints (1d) are start-up constraints: generator i is started up at node n if it is off at node n^- but on at node n. Constraints (1e) define the relationship between v_n^i and variables y_n^i and u_n^i. Note that in an optimal solution (with $\bar{S}^i > 0$ and $\underline{S}^i > 0$), we have $u_n^i = 0$ except the case when $y_{n^-}^i = 0$ and $y_n^i = 1$. Therefore, u_n^i will be forced to equal 1 only for the case $y_{n^-}^i = 0$ and $y_n^i = 1$. Constraints (1f) enforce the upper and lower bounds of power output by generator i if it is on at node n. Constraints (1g) ensure the load balance requirement. Constraints (1h) and (1i) are ramping constraints enforcing ramp-rate limits and start-up and shut-down ramp-rate limits of each generator i, as described in [10]. Finally, constraints (1j) are transmission capacity constraints as described in [31] and [27], and constraints (1k) indicate the binary unit commitment decisions. Note that we combine the uncertainties in the generation and load sides in this formulation, and regard the combined uncertainty as the net load uncertainty. In other words, we assume no curtailment of renewable energy, and consider renewable energy generation as the negative load in the power system. We make this assumption based on the fact that, in practice, the system operators are often required to use renewable energy as a priority over other resources (see, e.g., [8] and [30]). On the other hand, it can be observed that this formulation can be easily extended to explicitly incorporate the renewable energy by introducing an additional variable representing renewable energy quantity utilized with its upper bound defined as the renewable energy output. We simplify this formulation without loss of generality.

3 Strengthening the Minimum Up/Down Time Polytope

In this section, we derive strong valid inequalities for the minimum up/down time polytope. In particular, we provide a convex hull representation for the minimum up/down time constraints. The deterministic counterpart of this proof is shown in [22]. We generalize the result in [22] and extend it to a general stochastic scenario tree setting.

First, we note that constraints (1b)-(1e) allow $u_i^b = 1$ in cases (i) $y_{n^-}^i = y_n^i$ and (ii) $y_{n^-}^i = 1$ and $y_n^i = 0$. To ruling out case (i), we add the following constraint into (MSUC):

$$u_n^i \le \min\{y_n^i,\ 1 - y_{n^-}^i\}, \quad \forall n \in \mathcal{V} \setminus \{1\}, \forall i \in \mathcal{I}. \tag{2}$$

Considering constraints (1b)-(1e), (1k) and (2), we notice that (i) these constraints are decomposable to each generator i, and (ii) v can be represented by variables y and u. Hence, it is sufficient to analyze the following polytope for each generator $i \in \mathcal{I}$ by omitting the superscript i of y and u for notation brevity:

$$\begin{aligned} P := \Big\{(y,u) \in \mathbb{B}^{|\mathcal{V}|} \times \mathbb{B}^{|\mathcal{V}|-1} :\ & y_n - y_{n^-} \le y_i, \quad \forall n \in \mathcal{V} \setminus \{1\}, \forall i \in \mathcal{H}_{\bar{L}}(n), \\ & y_{n^-} - y_n \le 1 - y_i, \quad \forall n \in \mathcal{V} \setminus \{1\}, \forall i \in \mathcal{H}_{\underline{L}}(n), \\ & y_n - y_{n^-} \le u_n \le \min\{y_n,\ 1 - y_{n^-}\}, \quad \forall n \in \mathcal{V} \setminus \{1\}\Big\}. \end{aligned}$$

In the following, we focus on the convex hull of P, denoted as $\operatorname{conv}(P)$.

We propose the turn on/off inequalities on the scenario tree as follows:

$$\text{Turn on inequality:} \qquad \sum_{i=0}^{\bar{L}-1} u_{n_i^-} \le y_n, \qquad \forall n, \text{ such that } t(n) = \bar{L}+1, \ldots, T, \tag{3}$$

$$\text{Turn off inequality:} \qquad \sum_{i=0}^{\underline{L}-1} u_{n_i^-} \le 1 - y_{n_{\underline{L}}^-}, \qquad \forall n, \text{ such that } t(n) = \underline{L}+1, \ldots, T, \tag{4}$$

where n_j^- represents the j-fold ancestor of node n, i.e., $n_j^- = \{m \in \mathcal{P}(n) : t(m) = t(n) - j\}$. For example, $n_0^- = n$ and $n_1^- = n^-$. We show that the turn on/off inequalities are (i) valid for conv(P), and (ii) sufficient to describe conv(P) together with some trivial inequalities. We state the following claim and readers are referred to Appendix A for the detailed proofs.

Theorem 1 *The turn on/off inequalities* (3) *and* (4) *are valid for conv*(P). *Furthermore, we have conv*$(P) = Q$, *where*

$$Q = \Bigg\{ (y,u) \in \mathbb{R}_+^{|\mathcal{V}|} \times \mathbb{R}_+^{|\mathcal{V}|-1} : \begin{array}{l} \displaystyle\sum_{i=0}^{\bar{L}-1} u_{n_i^-} \le y_n, \forall n, \text{ such that } t(n) = \bar{L}+1, \ldots, T, \\ \displaystyle\sum_{i=0}^{\underline{L}-1} u_{n_i^-} \le 1 - y_{n_{\underline{L}}^-}, \forall n, \text{ such that } t(n) = \underline{L}+1, \ldots, T, \\ y_n - y_{n^-} \le u_n, \forall n \in \mathcal{V} \setminus \{1\} \Bigg\}. \end{array}$$

Proof: We first prove the validity of the turn on/off inequalities for conv(P), which implies that conv$(P) \subseteq Q$. On the other hand, the claim $Q \subseteq$ conv(P) follows from the following two claims. The detailed proofs are shown in Appendix A.

Claim 1 All the extreme points of Q are integral.

Claim 2 Any integral point in Q is also in P. ∎

Note here that the number of constraints in Q is a polynomial function of the input size of the scenario tree. Therefore, in the implementation of MSUC problem we can conveniently replace the minimum up/down time constraints with the constraints described in Q.

4 Strengthening the Ramping Polytope

In this section, we extend our study by incorporating the ramping constraints into P. For each leaf node n of $\mathcal{V}$ and each generator $i \in \mathcal{I}$, we let

$$Y_n^i = \Big\{(y, u, x) \in P \times \mathbb{R}_+^T : \quad \text{(1f), (1h), and (1i) are enforced}\Big\}. \tag{5}$$

In view that all the nodes considered in $\text{conv}(Y_n^i)$ belong to a particular scenario path, we can distinguish them by the time periods, e.g., we denote y_n as y_t where $t(n) = t$ and neglect the superscript i for notation brevity. We first observe that

(i) $\bar{Q} > \underline{Q} > 0, V > 0$ and $B > 0,$ and (6)

(ii) $\bar{V} \in [\underline{Q}, \bar{Q}]$ and $\bar{B} \in [\underline{Q}, \bar{Q}]$. (7)

Similarly, we state the following claims and readers are referred to Appendix B for the detailed proofs.

Proposition 1 *The ramping polytope conv*(Y_n^i) *is full-dimensional.*

Proof: See Appendix B.1 for the detailed proof. ■

In the remaining part of this section, we first propose two classes of extended ramping inequalities to strengthen the formulation describing ramping polytope $\text{conv}(Y_n^i)$ in Section 4.1, and thereafter we lift the extended ramping inequalities to obtain an even stronger valid inequalities in Section 4.2.

4.1 Extended Ramping Inequalities

For the two classes of extended ramping inequalities, the first one bounds the generation quantity for a single time period (see Propositions 2 and 3). For notation brevity, we let $[a, b]_{\mathbb{Z}}$ represent $[a, b] \cap \mathbb{Z}$, i.e., $\{a, a+1, \ldots, b\}$ for integers a and b.

Proposition 2 *For* $t \in [2, T]_{\mathbb{Z}}$*, the inequality*

$$x_{t-1} - \bar{B}y_{t-1} - (\bar{Q} - \bar{B})y_t \leq 0 \tag{8}$$

is valid for conv(Y_n^i)*.*

Proof: We discuss the following cases based on the values of y_{t-1} and y_t.

(i) If $y_{t-1} = y_t = 0$, inequality (8) is satisfied since $x_t = 0$ due to constraint (1f) in the definition of Y_n^i in (5).

(ii) If $y_{t-1} = 0$ and $y_t = 1$, inequality (8) is satisfied since $x_{t-1} = 0$ due to constraint (1f) and $\bar{Q} - \bar{B} \geq 0$ due to observation (7).

(iii) If $y_{t-1} = 1$ and $y_t = 0$, inequality (8) reduces to ramping-down constraint (1i) in the definition of Y_n^i in (5).

(iv) If $y_{t-1} = y_t = 1$, inequality (8) reduces to constraint (1f) in the definition of Y_n^i in (5). ■

We observe that inequality (8) bounds the generation quantity by considering a ramp-down case. Symmetrically, the inequality considering a ramp-up case can be shown as follows without proof.

Proposition 3 *For $t \in [2, T]_{\mathbb{Z}}$, the inequality*

$$x_t - \bar{V} y_t - (\bar{Q} - \bar{V}) y_{t-1} \leq 0 \tag{9}$$

is valid for conv(Y_n^i).

The second class of extended ramping inequalities bounds the difference of generation quantity in two consecutive time periods (see Propositions 4 and 5).

Proposition 4 *For $t \in [2, T]_{\mathbb{Z}}$, the inequality*

$$x_{t-1} - x_t - \bar{B} y_{t-1} + \min\{\underline{Q}, \bar{B} - B\} y_t \leq 0 \tag{10}$$

is valid for conv(Y_n^i). Furthermore, it is facet-defining if $\underline{Q} = \bar{B} - B$, $\bar{V} \geq \bar{B}$, and $\bar{Q} > \bar{B} + B$.

Proof: See Appendix B.2 for the detailed proof. ■

Again, by considering symmetry, the following three conclusions in Proposition 5 hold and are presented without proof.

Proposition 5 *For $t \in [2, T]_{\mathbb{Z}}$, the inequalities*

$$x_{t-1} - x_t + \underline{Q} y_t - \max\{\bar{B}, \underline{Q} + B\} y_{t-1} \leq 0, \tag{11}$$

$$x_t - x_{t-1} - \bar{V} y_t + \min\{\underline{Q}, \bar{V} - V\} y_{t-1} \leq 0, \tag{12}$$

$$\text{and} \quad x_t - x_{t-1} + \underline{Q} y_{t-1} - \max\{\bar{V}, \underline{Q} + V\} y_t \leq 0 \tag{13}$$

are valid for conv(Y_n^i). Furthermore, inequality (11) *is facet-defining for conv(Y_n^i) if $\underline{Q} = \bar{B} - B$, $\bar{V} \geq \bar{B}$, and $\bar{Q} > \bar{B} + B$. Besides, inequalities* (12) *and* (13) *are facet-defining for conv(Y_n^i) if $\underline{Q} = \bar{V} - V$, $\bar{B} \geq \bar{V}$, and $\bar{Q} > \bar{B} + B$.*

Observation: The facet-defining conditions for inequalities (10)-(13) indicate that inequalities (10) and (11) are equivalent when the facet-defining conditions $\underline{Q} = \bar{B} - B$, $\bar{V} \geq \bar{B}$, and $\bar{Q} > \bar{B} + B$ hold, and inequalities (12) and (13) are equivalent when the facet-defining conditions $\underline{Q} = \bar{V} - V$, $\bar{B} \geq \bar{V}$, and $\bar{Q} > \bar{B} + B$ hold. However, they are different when the facet-defining conditions are not satisfied.

4.2 Lifted Ramping Inequalities

In this subsection, we lift the extended ramping inequalities (8) and (9) to obtain further stronger valid inequalities for conv(Y_n^i). Inequalities (8) and (9) only consider two consecutive time periods and only include variables x and y. In this part, we generalize these inequalities by considering more consecutive time periods and utilize the minimum up/down time polytope by introducing variable u in the inequalities through the lifting procedure. More importantly, this procedure maintains the sequence independent property due to the problem structure.

First, we generalize inequality (8) by considering more consecutive time periods and obtain a seed inequality with the corresponding variable u fixed at zero. We define constants $K_1 = \max\{n \in \mathbb{Z} : \bar{V} + nV < \bar{Q}\}$, $K_2 = \max\{n \in \mathbb{Z} : \bar{B} + nB < \bar{Q}\}$, and $K = \min\{K_2 + 1, \bar{L} - 1, \underline{L}\}$. Note here that $K_1, K_2 \geq -1$ due to observation (7), and $K \geq 0$ since $K_2 \geq -1$ and $\bar{L} \geq 1$. We also define $a^+ = \max\{a, 0\}$ for any $a \in \mathbb{R}$.

Lemma 1 *For $t \in [\bar{L} - K + 1, T - K]_{\mathbb{Z}}$, the seed inequality*

$$x_t \leq \bar{B} y_t + \sum_{i=1}^{K-1} B y_{t+i} + \Big(\bar{Q} - \bar{B} - (K-1)^+ B\Big) y_{t+K} \tag{14}$$

is valid for the lower-dimensional space of $conv(Y_n^i)$ *by fixing* $u_{t+i} = 0$, $\forall i \in W := \big[K - \bar{L} + 1, \min\{T - t, \underline{L}\}\big]_{\mathbb{Z}}$*, and facet-defining if* $K = K_2 + 1$ *and* $\bar{V} + (\bar{L} - K)V \geq \bar{B} + K_2 B$*.*

Proof: See Appendix B.3 for the detailed proof. ■

Then, we lift seed inequality (14) by reintroducing variables $\{u_{t+i}, i \in W\}$, where $W = \big[K - \bar{L} + 1, \min\{T - t, \underline{L}\}\big]_{\mathbb{Z}}$. As indicated before, we find that lifting these variables is convenient because it possesses the desired sequence independent property, as the following proposition states.

Theorem 2 *For* $t \in [\bar{L} - K + 1, T - K]_{\mathbb{Z}}$*, variables* $\{u_{t+i}, i \in W\}$ *can be lifted in a sequence independent procedure from seed inequality* (14)*, i.e., lifting variable* u_{t+i} *is independent of any other variable* $u_{t+i'}$ *for* $i, i' \in W$ *and* $i' \neq i$*.*

Proof: Since we include the minimum up/down time constraints in the ramping polytope $\mathrm{conv}(Y_n^i)$, we have

$$
\begin{aligned}
\sum_{i \in W} u_{t+i} &= \sum_{i=K-\bar{L}+1}^{\min\{T-t,\underline{L}\}} u_{t+i} \\
&= \sum_{i=K-\bar{L}+1}^{0} u_{t+i} + \sum_{i=1}^{\min\{T-t,\underline{L}\}} u_{t+i} \\
&\leq \sum_{i=-\bar{L}+1}^{0} u_{t+i} + \sum_{i=1}^{\min\{T-t,\underline{L}\}} u_{t+i} \\
&\leq y_t + (1 - y_t) = 1,
\end{aligned}
$$

where the first inequality follows from $K \geq 0$ and the second inequality follows from the turn on/off inequalities. Hence, $u_{t+i} = 1$ implies that $u_{t+i'} = 0$, $\forall i' \in W$ and $i' \neq i$. It follows that when we lift any variable u_{t+i} for $i \in W$, all the other variables $u_{t+i'}$ with $i' \in W$ and $i' \neq i$ are automatically fixed to be zero. The sequence independent property follows since neither the lifting coefficient nor the value of $u_{t+i'}$ influences the lifting procedure of u_{t+i}. ■

By applying the sequence independent lifting property described in Theorem 2, the following conclusion holds.

Proposition 6 *For $t \in [\bar{L} - K + 1, T - K]_{\mathbb{Z}}$, the lifted ramping inequality*

$$\begin{aligned} x_t \leq \; & \bar{B} y_t + \sum_{i=1}^{K-1} B y_{t+i} + \Big(\bar{Q} - \bar{B} - (K-1)^+ B\Big) y_{t+K} + \\ & \sum_{i=\max\{-K_1, K-\bar{L}+1\}}^{0} (\bar{V} - iV - \bar{Q}) u_{t+i} + \sum_{i=1}^{K} \Big(\bar{B} + (i-1)B - \bar{Q}\Big) u_{t+i} \end{aligned} \tag{15}$$

is valid for conv(Y_n^i), and is facet-defining if $K = K_2 + 1$ and $\bar{V} + (\bar{L} - K)V \geq \bar{B} + K_2 B$.

Proof: **(Validity)** We lift variables $\{u_{t+i}, i \in W\}$ in seed inequality (14). We let π_{t+i} represent the desired lifting coefficient of each variable u_{t+i}, and hence the lifted ramping inequality is in the form

$$x_t \leq \bar{B} y_t + \sum_{i=1}^{K-1} B y_{t+i} + \Big(\bar{Q} - \bar{B} - (K-1)^+ B\Big) y_{t+K} + \sum_{i=K-\bar{L}+1}^{\min\{T-t, \underline{L}\}} \pi_{t+i} u_{t+i}. \tag{16}$$

Now we derive the value for each π_{i+i}. By lifting the variable u_{t+i} from zero to one, we discuss the following cases:

(i) If $i \in [K - \bar{L} + 1, 0]_{\mathbb{Z}}$, then $y_{t+i'} = 1$, $\forall i' \in [0, K]_{\mathbb{Z}}$ by the turn on inequality since $i' - i + 1 \leq K - (K - \bar{L} + 1) + 1 = \bar{L}$. Hence, by inequality (16) we have $x_t \leq \bar{Q} + \pi_{t+i}$. But since we start up the generator at period $t + i$, we have $x_t \leq \min\{\bar{Q}, \bar{V} - iV\}$. It follows that π_{t+i} should be chosen to be $\min\{\bar{Q}, \bar{V} - iV\} - \bar{Q}$. That is, by the definition of K_1,

$$\pi_{t+i} = \begin{cases} \bar{V} - iV - \bar{Q}, & \forall i \geq -K_1, \\ 0, & \text{o.w.} \end{cases}.$$

(ii) If $i \in [1, K]_{\mathbb{Z}}$, then $y_{t+i'} = 0$ for each $i' \in [0, i-1]_{\mathbb{Z}}$ and $y_{t+i'} = 1$ for each $i' \in [i, K]_{\mathbb{Z}}$ by the turn on/off inequalities since $K \leq \min\{\bar{L} - 1, \underline{L}\}$. Hence, by inequality (16) we have $x_t \leq \bar{Q} - \bar{B} - (i-1)B + \pi_{t+i}$. But since the generator is off at time period t, we have $x_t \leq 0$. It follows that π_{t+i} should be chosen to be $\bar{B} + (i-1)B - \bar{Q}$.

(iii) If $i \in [K + 1, \min\{T - t, \underline{L}\}]_{\mathbb{Z}}$, then $y_{t+i'} = 0$ for each $i' \in [0, K]_{\mathbb{Z}}$ by the turn off inequality since $i - i' \leq \min\{T - t, \underline{L}\} \leq \underline{L}$. Hence, by inequality (16) we have $x_t \leq \pi_{t+i}$. But since the generator is off at time period t, we have $x_t \leq 0$. It follows that π_{t+i} should be chosen to be 0.

The proof of validity is complete by substituting the values of π_{t+i} in inequality (16).

(Facet-defining) We note that the seed inequality (14) is facet-defining under the conditions $K = K_2 + 1$ and $\bar{V} + (\bar{L} - K)V \geq \bar{B} + K_2 B$. It follows from the lifting theory (see, e.g., [9]) that the lifted ramping inequality is facet-defining for $\text{conv}(Y_n^i)$. ■

Similarly, we can obtain a strong valid inequality for $\text{conv}(Y_n^i)$ by generalizing the lifting inequality (9). Symmetrical to the argument we conduct in developing the lifted ramping inequality (15), we now consider first fixing the variables v and lifting them back afterwards. The proof is similar as well and the following result is sated without proof.

Proposition 7 *For $t \in [K' + 1, T + K' - \bar{L}]_{\mathbb{Z}}$, the inequality*

$$\begin{aligned} x_t \leq \; & \bar{V} y_t + \sum_{i=1}^{K'-1} V y_{t-i} + \Big(\bar{Q} - \bar{V} - (K'-1)^+ V\Big) y_{t-K'} + \\ & \sum_{i=-K'+1}^{0} (\bar{V} - iV - \bar{Q}) v_{t+i} + \sum_{i=1}^{\min\{-K'+\bar{L}, K_2+1\}} \Big(\bar{B} + (i-1)B - \bar{Q}\Big) v_{t+i} \end{aligned} \tag{17}$$

with $v_{t+i} = y_{t-1} - y_t + u_t$, $\forall i \in [-K' + 1, -K' + \bar{L}]_{\mathbb{Z}}$ is valid for $\text{conv}(Y_n^i)$, and is facet-defining if $K' = K_1 + 1$ and $\bar{B} + (\bar{L} - K')B \geq \bar{V} + K_1 V$, where $K' = \min\{K_1, \bar{L}, \underline{L} - 1\}$.

Finally, we note that the numbers of all the valid inequalities proposed in this section (i.e., inequalities (8)-(13), (15), (17)) are polynomial functions of the input size of the problem. That is, we can introduce all these inequalities at the beginning of the implementation to strengthen the ramping polytope without increasing the problem size too much.

5 Strengthening the Economic Dispatch Polytope

In this section, we study the economic dispatch polytope, i.e., the polytope generated by constraints (1f), (1g) and (1k). Since this polytope appears at each node $n \in \mathcal{V}$, we ignore the subscript n and accordingly

$$Z = \Big\{(y, x) \in \mathbb{B}^I \times \mathbb{R}_+^I : \quad \sum_{i=1}^{I} x_i \geq D, \; \underline{Q}_i y_i \leq x_i \leq \bar{Q}_i y_i, \forall i \in \mathcal{I}\Big\}.$$

Meanwhile, the strong valid inequalities obtained can be applied for each node in the scenario tree.

It is easy to observe that $\bar{Q}_i > \underline{Q}_i > 0$ for each $i \in \mathcal{I}$. Without loss of generality, we assume

$$\sum_{j \in \mathcal{I} \setminus \{i\}} \bar{Q}_j \geq D \text{ for each } i \in \mathcal{I} = \{1, \ldots, I\}, \tag{18}$$

which leads to the following claim.

Proposition 8 *The polytope conv(Z) is full-dimensional.*

Proof: See Appendix C.1 for the detailed proof. ∎

Now we develop strong valid inequalities for conv(Z). We start with finding seed inequalities for conv(Z) in the low-dimensional space by fixing a part of binary variables y_i to zero. Then, we lift the seed inequalities to obtain valid inequalities for conv(Z) in the original space. In our approach, we explicitly compute the corresponding lifting functions and discuss if they are subadditive. If a lifting function is subadditive, we can lift the seed inequality in a sequence independent manner. Otherwise, we can approximate the lifting function by a tight subadditive function, and again lift the seed inequality through a sequence independent lifting procedure. We also give sufficient conditions for the valid inequalities thus obtained to be facet-defining for conv(Z). More importantly, these lifted inequalities are in closed forms.

5.1 Seed Inequalities

In this subsection, we introduce three classes of cover inequalities for conv(Z) in the low-dimensional space by fixing a part of binary variables y_i (and hence x_i) to zero, which are used as seed inequalities for the following subsection. For a given set $C \subseteq \mathcal{I}$, we define

$$Z_C = \mathrm{proj}_{\mathbb{B}^{|C|} \times \mathbb{R}_+^{|C|}} \Big\{ (y, x) \in Z : \quad y_i = 0, \ \forall i \in \mathcal{I} \setminus C \Big\},$$

where C is called a cover. Based on this cover definition, the following cover inequalities are valid for conv(Z_C).

Lemma 2 *For a given cover $C = C_1 \cup C_2$ such that $C_1 \cap C_2 = \emptyset$ and $\sum_{i \in C_1} \bar{Q}_i + \sum_{i \in C_2} \underline{Q}_i < D \leq \sum_{i \in C} \bar{Q}_i$, we let $\Delta = D - (\sum_{i \in C_1} \bar{Q}_i + \sum_{i \in C_2} \underline{Q}_i)$, and the following cover inequality*

$$\sum_{i \in C_2} (x_i - \underline{Q}_i y_i) \geq \Delta \tag{19}$$

is valid for conv(Z_C).

Proof: For any given point $(x, y) \in Z_C$, we have $x_i = 0$ because $y_i = 0$ for each $i \in \mathcal{I} \setminus C$. Then, we have

$$D \leq \sum_{i \in C} x_i = \sum_{i \in C_1} x_i + \sum_{i \in C_2} (x_i - \underline{Q}_i y_i + \underline{Q}_i y_i) \leq \sum_{i \in C_1} \bar{Q}_i + \sum_{i \in C_2} (x_i - \underline{Q}_i y_i) + \sum_{i \in C_2} \underline{Q}_i,$$

where the first inequality follows from the condition $D \leq \sum_{i \in C} \bar{Q}_i$, and the second inequality follows from the fact that $x_i \leq \bar{Q}_i$ for each $i \in C_1$ and $y_i \leq 1$ for each $i \in C_2$. Therefore, (19) is satisfied due to the definition of Δ. ■

Lemma 3 *For a given cover C such that $\sum_{i \in C} \underline{Q}_i < D < \sum_{i \in C} \bar{Q}_i$, we let $\Gamma = D - \sum_{i \in C} \underline{Q}_i$, and the following cover inequality*

$$\sum_{i \in C} \Big(x_i - (\underline{Q}_i - 1) y_i \Big) \geq \Gamma + |C| \tag{20}$$

is valid for conv(Z_C).

Proof: For any given point $(x, y) \in Z_C$, we let $T = \{i \in C : y_i = 0\}$. Accordingly, we have

$$\begin{aligned} D \quad &\leq \quad \sum_{i \in C} x_i = \sum_{i \in C} \underline{Q}_i y_i + \sum_{i \in C} (x_i - \underline{Q}_i y_i) \\ &\leq \quad \sum_{i \in C \setminus T} \underline{Q}_i + \sum_{i \in C} (x_i - \underline{Q}_i y_i), \end{aligned}$$

where the first inequality follows from the condition $D < \sum_{i \in C} \bar{Q}_i$, and the second inequality follows from the fact that $y_i = 0$ for each $i \in T$ and $y_i = 1$ for each $i \in C \setminus T$. It follows from the definition of Γ that $\sum_{i \in C} (x_i - \underline{Q}_i y_i) \geq \Gamma + \sum_{i \in T} \underline{Q}_i$. Therefore, we have

$$\sum_{i \in C} \Big(x_i - (\underline{Q}_i - 1) y_i \Big) = \sum_{i \in C} (x_i - \underline{Q}_i y_i) + \sum_{i \in C} y_i \geq \Gamma + \sum_{i \in T} \underline{Q}_i + |C| - |T| \geq \Gamma + |C|,$$

where the first inequality is due to $y_i = 0$ for each $i \in T$ and $y_i = 1$ for each $i \in C \setminus T$, and the second inequality follows from the fact that $\underline{Q}_i \geq 1$ for each $i \in T$. ■

Lemma 4 *For a given cover C such that $\sum_{i \in C} \bar{Q}_i \geq D$, the cover inequality*

$$\sum_{i \in C} y_i \geq R \tag{21}$$

is valid for conv(Z_C)*, where* R *is a nonnegative integer satisfying the condition* $\sum_{i=|C|-R+2}^{|C|} \bar{Q}_{[i]} < D \leq \sum_{i=|C|-R+1}^{|C|} \bar{Q}_{[i]}$*, and* $\bar{Q}_{[1]} \leq \cdots \leq \bar{Q}_{[|C|]}$ *is a sorted nondecreasing order of* $\{\bar{Q}_i : i \in C\}$*. Furthermore, cover inequality* (21) *is facet-defining for conv*(Z_C) *if* $R = |C| - 1$ *and* $D < \sum_{i=1}^{|C|-1} \bar{Q}_{[i]}$*.*

Proof: **(Validity)** Proof by contradiction. For a given point $(x, y) \in Z_C$, if $\sum_{i\in C} y_i \leq R - 1$, then

$$\sum_{i=1}^{I} x_i = \sum_{i\in C} x_i \leq \sum_{i\in C} \bar{Q}_i y_i \leq \sum_{i=|C|-R+2}^{|C|} \bar{Q}_{[i]} < D,$$

where the second inequality follows from the sorted order definition of $\bar{Q}_{[i]}$ for $i \in C$. This contradicts with the fact that $\sum_{i=1}^{I} x_i \geq D$. Therefore, the original condition holds.

(Facet-defining) We prove by generating affinely independent points. Detailed proof is shown in Appendix C.2. ■

5.2 Lifted Strong Inequalities

In this subsection, we lift cover inequalities (19), (20) and (21). For inequality (19), we obtain a subadditive lifting function and so the lifting procedure is sequence independent (see, e.g., [33] and [9]). For inequality (20), although the lifting function is not subadditive in general, the subadditivity property does hold under mild assumptions. For inequality (21), the lifting function is not subadditive in general, and we propose approximation lifting and identify the conditions under which this inequality is facet-defining.

In our approach, we lift unit commitment status (i.e., variable y) and generation quantity (i.e., variable x) simultaneously each time, which is different from the traditional lifting procedures like the one described in Section 4.2. To the best of our knowledge, our lifting approach is most similar to the procedure by [23]. For a given valid inequality

$$\sum_{i\in C\cup C^j} \alpha_i y_i + \sum_{i\in C\cup C^j} \beta_i x_i \geq \gamma \tag{22}$$

for conv$(Z_{C\cup C^j})$ where C is a cover in $\mathcal{I}$ and $C^j \subseteq \mathcal{I} \setminus C$ and $|C^j| = j$ with $C^0 = \emptyset$, the lifting function is

$$F^j(z) = \gamma - \min_{y,x} \Big\{ \sum_{i\in C\cup C^j} (\alpha_i y_i + \beta_i x_i) \Big\} \tag{23a}$$

$$\text{s.t.} \sum_{i\in C\cup C^j} x_i \geq D - z, \tag{23b}$$

$$\underline{Q}_i y_i \leq x_i \leq \bar{Q}_i y_i, \quad \forall i \in C \cup C^j, \tag{23c}$$

$$y_i \in \{0,1\}, \quad \forall i \in C \cup C^j. \tag{23d}$$

To lift the pair (y_k, x_k) for some $k \in \mathcal{I} \setminus \{C \cup C^j\}$, if there exists a pair $(\alpha_k, \beta_k) \in \mathbb{R}^2$ such that

$$\alpha_k + \beta_k x_k \geq F^j(x_k), \quad \forall x_k \in [\underline{Q}_k, \bar{Q}_k],$$

then the inequality

$$\sum_{i\in C\cup C^{j+1}} \alpha_i y_i + \sum_{i\in C\cup C^{j+1}} \beta_i x_i \geq \gamma \tag{24}$$

is valid for $\text{conv}(Z_{C\cup C^{j+1}})$ where $C^{j+1} = C^j \cup \{k\}$.

Along this approach, we can obtain the following sequence independent lifting property.

Lemma 5 *Given a valid inequality* (22) *with* $j = 0$ *for conv*(Z_C)*, if (i)* $F^{|C|}(z)$ *is subadditive over* $\mathbb{R}_+$ *and (ii) there exists* $(\alpha_i, \beta_i) \in \mathbb{R}^2$ *for all* $i \in \mathcal{I} \setminus C$ *such that*

$$\alpha_i + \beta_i x_i \geq F^{|C|}(x_i), \quad \forall x_i \in [\underline{Q}_i, \bar{Q}_i], \tag{25}$$

then the inequality

$$\sum_{i\in\mathcal{I}} \alpha_i y_i + \sum_{i\in\mathcal{I}} \beta_i x_i \geq \gamma \tag{26}$$

is valid for conv(Z)*. Further, if (i) inequality* (22) *is facet-defining for conv*(Z_C)*, (ii) conv*(Z_C) *is full-dimensional and (iii) coefficients* (α_i, β_i) *are chosen in a way that two linearly independent points in conv*(Z) *satisfy* (25) *at equality, then* (26) *is facet-defining for conv*(Z)*.*

In the remainder of this subsection, we derive strong valid inequalities based on seed inequalities (19), (20), and (21).

Theorem 3 *For a given cover seed inequality* (19)*, the lifting function is*

$$F_1(z) = \begin{cases} -\infty, & \text{if } z \in \left(-\infty, D - \sum_{i\in C} \bar{Q}_i\right), \\ z, & \text{if } z \in \left[D - \sum_{i\in C} \bar{Q}_i, \Delta\right), \\ \Delta, & \text{if } z \in [\Delta, \infty), \end{cases} \tag{27}$$

which is subadditive on $\mathbb{R}_+$ *(see Figure 2 for illustration).*

Proof: We derive lifting function $F_1(z)$ by solving the embedded optimization problem (23) and prove the subadditivity by showing that $F_1(z_1 + z_2) \leq F_1(z_1) + F_1(z_2)$ for any $z_1, z_2 \in \mathbb{R}_+$. The detailed proof is shown in Appendix C.3. ■

In general, inequality (19) is not facet-defining for conv(Z_C). However, this inequality can be

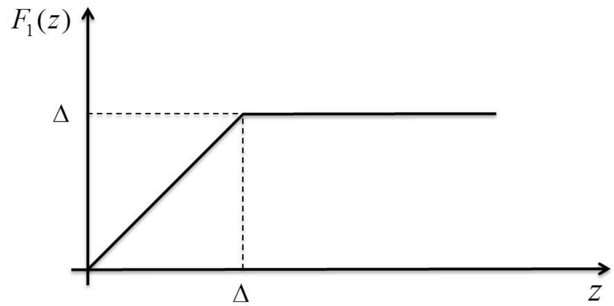

Figure 2: Lifting function $F_1(z)$ on $\mathbb{R}_+$

strengthened to be facet-defining after the lifting procedure as shown in the following conclusion.

Proposition 9 *For a given cover seed inequality* (19)*, the lifted inequality*

$$\sum_{i \in C_2} (x_i - \underline{Q}_i y_i) + \sum_{i \in \mathcal{I} \setminus C} (\alpha_i y_i + \beta_i x_i) \geq \Delta, \tag{28}$$

$$\text{where} \;\; (\alpha_i, \beta_i) = \begin{cases} (\Delta, 0), & \text{if } \Delta \leq \underline{Q}_i \\ (\Delta, 0) \text{ or } (0,1), & \text{if } \underline{Q}_i < \Delta < \bar{Q}_i \\ (0,1), & \text{if } \Delta \geq \bar{Q}_i \end{cases}, \;\; \forall i \in \mathcal{I} \setminus C, \tag{29}$$

is valid for conv(Z)*. Furthermore, it is facet-defining for conv*(Z) *if*

(i) $\underline{Q}_i + \Delta \leq \bar{Q}_i$, $\forall i \in C_2$;

(ii) $\exists s \in \mathcal{I} \setminus C$, *such that* $\bar{Q}_s \geq \bar{Q}_i + \Delta$, $\forall i \in C_1$, *and* $\bar{Q}_s \geq \underline{Q}_i + \Delta$, $\forall i \in C_2$;

(iii) $\Delta \neq \bar{Q}_i$, $\forall i \in \mathcal{I} \setminus (C \cup \{s\})$.

Proof: We prove the validity of lifted inequality (28) by following the sequence independent procedure described in Lemma 5, and the facet-defining property by generating affinely independent points satisfying inequality (28) at equality. The detailed proof is shown in Appendix C.4. ■

We now discuss how to find violated lifted cover inequalities (28) for a given point $(y, x) \in \mathbb{B}^I \times \mathbb{R}^I_+$. In view that the separation problem for cover inequalities for binary integer programs is

NP-hard in general (see, e.g., [14]), we provide a heuristic separation algorithm. The effectiveness of the proposed separation algorithm is verified by the computational experiments in Section 6. The similar claim holds for separating the lifted cover inequalities (32) and (35) we will derive later on.

Separation 1: We note that inequality (28) is equivalent to

$$\sum_{i\in C_1}\bar{Q}_i+\sum_{i\in C_2}\Big(x_i+\underline{Q}_i(1-y_i)\Big)+\sum_{i\in\mathcal{I}\setminus C}(\alpha_i y_i+\beta_i x_i)\geq D. \tag{30}$$

For a given point $(\hat{y},\hat{x})\in\mathbb{B}^I\times\mathbb{R}^I_+$, to separate (28), we try to find an inequality such that the left hand side of (30) is as small as possible. If $\hat{y}_i$ is close to 1, there is no much difference between $\bar{Q}_i$ and $\hat{x}_i+\underline{Q}_i(1-\hat{y}_i)$. Thus, we put the corresponding indices i into C_1. If $\hat{y}_i$ is positive but very small, e.g., $\hat{y}_i<0.1$, then $\hat{x}_i+\underline{Q}_i(1-\hat{y}_i)$ is close to $(\bar{Q}_i-\underline{Q}_i)\hat{y}_i+\underline{Q}_i$, assuming $\hat{x}_i=\bar{Q}_i\hat{y}_i$ in the linear program relaxation solution, which might be much smaller than $\bar{Q}_i$. Thus, we pre-select the corresponding indices i and put them into C_2. Now we sort the indices pre-selected into C_2 in a non-decreasing order according to the value $\hat{x}_i+\underline{Q}_i(1-\hat{y}_i)$, and meanwhile finalize the indices one by one based on this order until $\sum_{i\in C_1\cup C_2}\bar{Q}_i$ exceeds D for the first time with the remaining indices dropped (note here that the cover C is valid if $\sum_{i\in C_1}\bar{Q}_i+\sum_{i\in C_2}\underline{Q}_i<D\leq\sum_{i\in C}\bar{Q}_i$ based on Lemma 2). Next we check if $\sum_{i\in C_1}\bar{Q}_i+\sum_{i\in C_2}\underline{Q}_i<D$ for the validity of cover C. If not, then we stop with no inequality found; if yes, then we let $\Delta=D-\Big(\sum_{i\in C_1}\bar{Q}_i+\sum_{i\in C_2}\underline{Q}_i\Big)$ and generate lifting coefficients (α_i,β_i) for each $i\in\mathcal{I}\setminus C$ according to (29). Finally, we check if the lifted cover inequality (28) is violated: if not, then we stop with no inequality found; if yes, then we return the corresponding inequality (28), which cuts off the given point $(\hat{y},\hat{x})$.

Theorem 4 *For a given cover seed inequality* (20), $\underline{Q}_{[1]}\leq\underline{Q}_{[2]}\leq\cdots\leq\underline{Q}_{[|C|]}$ *is a nondecreasing sorted order of the set* $\{\underline{Q}_i:\ i\in C\}$, $A_j=\Gamma+\sum_{i=1}^{j}\underline{Q}_{[i]}$ *for each* $j=0,\ldots,|C|$, *and the lifting function is*

$$F_2(z)=\begin{cases}-\infty, & \text{if } z\in\big(-\infty,D-\sum_{i\in C}\bar{Q}_i\big)\\ z, & \text{if } z\in\big[D-\sum_{i\in C}\bar{Q}_i,A_0\big)\\ \Gamma+j, & \text{if } z\in[A_j,A_{j+1}-1),\ \forall j=0,\ldots,|C|-2\\ z-A_{j+1}+1+\Gamma+j, & \text{if } z\in[A_{j+1}-1,A_{j+1}),\ \forall j=0,\ldots,|C|-2\\ \Gamma+|C|-1, & \text{if } z\in[A_{|C|-1},A_{|C|})\\ \Gamma+|C|, & \text{if } z\in[A_{|C|},\infty)\end{cases},$$

We call this function "almost" subadditive over $\mathbb{R}_+$ because a related function

$$\hat{F}_2(z) = \begin{cases} z - A_{|C|} + \Gamma + |C|, & \text{if } z \in [A_{|C|} - 1, A_{|C|}) \\ F_2(z), & o.w. \end{cases}$$

is subadditive over $\mathbb{R}_+$, and the two functions do not match only in a small interval with length less than 1 (see Figure 3 for comparison).

Proof: We derive lifting function $F_2(z)$ by solving the embedded optimization problem (23) and prove the subadditivity by showing that $F_2(z_1 + z_2) \leq F_2(z_1) + F_2(z_2)$ for any $z_1, z_2 \in \mathbb{R}_+$. The detailed proof is shown in Appendix C.5. ■

To obtain the lifting coefficients for inequality (20), we need to find piecewise linear functions that

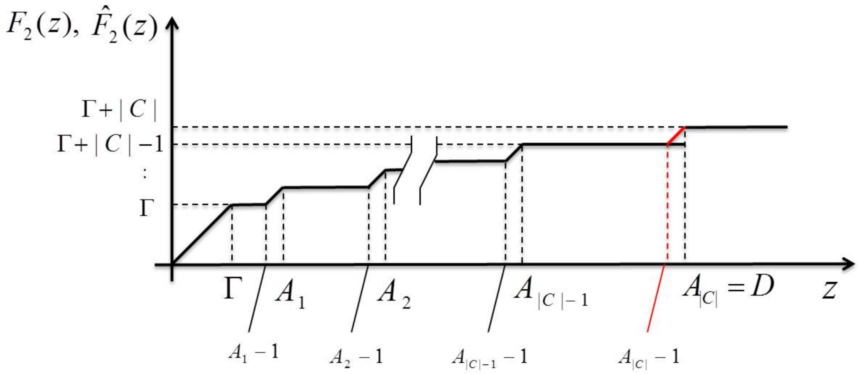

Figure 3: Lifting functions $F_2(z)$ and $\hat{F}_2(z)$ on $\mathbb{R}_+$, where $\hat{F}_2(z)$ is marked in red in interval $[A_{|C|} - 1, A_{|C|})$.

overestimate the lifting function $\hat{F}_2(z)$ over the interval $[\underline{Q}_i, \bar{Q}_i]$ for each $i \in C$, and touch $F_2(z)$ at two points. To that end, we construct a tight piecewise linear overestimation of $\hat{F}_2(z)$ as follows:

Lemma 6 *For each $i \in \mathcal{I} \setminus C$, define set $\Lambda_i = \{A_j : \underline{Q}_i \leq A_j \leq \bar{Q}_i, j \in C \cup \{0\}\} \cup \{\underline{Q}_i, \bar{Q}_i\}$, and let $B_i^1 < B_i^2 < \cdots < B_i^{|\Lambda_i|}$ be its increasing sorted order. Then the piecewise linear function*

$$\Phi_i(z) = \min_{k=1,\ldots,|\Lambda_i|-1} \{\phi_i^k(z)\}, \tag{31}$$

where $\phi_i^k(z) = \hat{F}_2(B_i^k) + \left(\frac{\hat{F}_2(B_i^{k+1}) - \hat{F}_2(B_i^k)}{B_i^{k+1} - B_i^k}\right)(z - B_i^k)$ is an overestimation of $\hat{F}_2(z)$ on $[\underline{Q}_i, \bar{Q}_i]$, and each $\phi_i^k(z)$ touches $\hat{F}_2(z)$ at two points $(B_i^k, \hat{F}_2(B_i^k))$ and $(B_i^{k+1}, \hat{F}_2(B_i^{k+1}))$.

Proof: The detailed proof is shown in Appendix C.6. ■

With the piecewise linear overestimation of the lifting function defined in Lemma 6, we can lift the cover inequality (20) as follows by skipping the mismatch interval $[A_{|C|}-1, A_{|C|})$ through an assumption $\bar{Q}_i \leq D-1$ for all $i \in \mathcal{I}$. Note here that $A_{|C|} = \Gamma + \sum_{k=1}^{|C|} Q_{[k]} = D$ by the definition of $A_{|C|}$.

Proposition 10 *If $\bar{Q}_i \leq D-1$ for all $i \in \mathcal{I}$, then the inequality*

$$\sum_{i \in C}\Big(x_i - (\underline{Q}_i - 1)y_i\Big) + \sum_{i \in \mathcal{I}\setminus C} (\alpha_i y_i + \beta_i x_i) \geq \Gamma + |C| \tag{32}$$

is valid for conv(Z), where (α_i, β_i) is in the form of

$$\left(\frac{F_2(B_i^k)B_i^{k+1} - F_2(B_i^{k+1})B_i^k}{B_i^{k+1} - B_i^k}, \quad \frac{F_2(B_i^{k+1}) - F_2(B_i^k)}{B_i^{k+1} - B_i^k}\right) \tag{33}$$

corresponding to each $k = 1, \ldots, |\Lambda_i| - 1$. Furthermore, inequality (32) *defines a face of conv(Z) of dimension at least $2I - |T| - |C| - 1$, where $T = \{i \in \mathcal{I} \setminus C : \bar{Q}_i \leq \Gamma\}$.*

Proof: The detailed proof is shown in Appendix C.7. ■

Separation 2: We note that inequality (32) is equivalent to

$$\sum_{i \in C}\Big(x_i + y_i + \underline{Q}_i(1 - y_i)\Big) - |C| + \sum_{i \in \mathcal{I}\setminus C} (\alpha_i y_i + \beta_i x_i) \geq D.$$

For a given point $(\hat{y}, \hat{x}) \in \mathbb{B}^I \times \mathbb{R}_+^I$, we first follow a similar greedy approach as described in Separation 1 to initiate a cover C such that $D < \sum_{i \in C} \bar{Q}_i$. The only difference is that we need to consider the value $\hat{x}_i + \hat{y}_i + \underline{Q}_i(1-\hat{y}_i)$ instead of $\hat{x}_i + \underline{Q}_i(1-\hat{y}_i)$ in Separation 1. Similarly, we check if $\sum_{i \in C} \underline{Q}_i < D$ for the validity of the initial cover C. If not, then we stop with no inequalities found; if yes, then we let $\Gamma = D - \sum_{i \in C} \underline{Q}_i$ and generate lifting coefficients (α_i, β_i) for each $i \in \mathcal{I} \setminus C$ according to (33). Note that there can be multiple choices of (α_i, β_i) due to different possible values of $k \in \{1, \ldots, |\Lambda_i| - 1\}$. To find the (α_i, β_i) that provides the tightest inequality, we find the interval $[B_i^k, B_i^{k+1})$, $k \in \{1, \ldots, |\Lambda_i| - 1\}$, such that $B_i^k \leq \hat{x}_i < B_i^{k+1}$, with the exception that $\hat{x}_i < B_i^1$ (i.e., $\hat{x}_i < \underline{Q}_i$) for which we select $[B_i^1, B_i^2)$. Finally, we check if the lifted cover inequality (32) is violated: if not, then we stop with no inequality found; if yes, then we return the corresponding inequality (32), which cuts off the given point $(\hat{y}, \hat{x})$.

Now we lift the cover inequality (21) of strong version, i.e., with $R = |C| - 1$ and $D < \sum_{i=1}^{|C|-1} \bar{Q}_{[i]}$.

Proposition 11 *For a cover seed inequality* (21) *where* $R = |C| - 1$, $D < \sum_{i=1}^{|C|-1} \bar{Q}_{[i]}$, *we let* $\Omega = D - \sum_{i=3}^{|C|} \bar{Q}_{[i]}$ *and* $G_j = \sum_{i=3}^{j} \bar{Q}_{[i]}$ *for* $j = 2, \ldots, |C|$. *The lifting function is*

$$F_3(z) = \begin{cases} -\infty, & \text{if } z \in (-\infty, D - \sum_{i \in C} \bar{Q}_i), \\ -1, & \text{if } z \in [D - \sum_{i \in C} \bar{Q}_i, D - \sum_{i=2}^{|C|} \bar{Q}_{[i]}), \\ 0, & \text{if } z \in [D - \sum_{i=2}^{|C|} \bar{Q}_{[i]}, \Omega), \\ j-1, & \text{if } z \in [\Omega + G_j, \Omega + G_{j+1}),\ \forall j = 2, \ldots, |C|-1, \\ |C|-1, & \text{if } z \in [\Omega + G_{|C|}, \infty). \end{cases}$$

Proof: The detailed proof is shown in Appendix C.8. ■

Unlike the previous two lifting functions, $F_3(z)$ is not subadditive in general. To recapture subadditivity, we apply the approximate lifting function recently developed in [6].

Proposition 12 *(Proposition 3.23 in [6]) The function*

$$\tilde{F}_3(z) = \begin{cases} \frac{z}{\Omega}, & \text{if } z \in [0, \Omega) \\ j-1, & \text{if } z \in [\Omega + G_j, G_{j+1}),\ \forall j = 2, \ldots, |C|-1, \\ j-1+\frac{z-G_{j+1}}{\Omega}, & \text{if } z \in [G_{j+1}, \Omega + G_{j+1}),\ \forall j = 2, \ldots, |C|-1, \\ |C|-1, & \text{if } z \in [\Omega + G_{|C|}, \infty) \end{cases}$$

is a valid subadditive approximation of $F_3(z)$ *that is nondominated over* $\mathbb{R}_+$, *where* $\tilde{F}_3(z)$ *being nondominated is defined that there does not exist a valid subadditive approximation* $J(z)$ *of* $F_3(z)$, *such that* $J(z) < \tilde{F}_3(z)$ *for some* $z \in \mathbb{R}_+$ *(see Figure 4).*

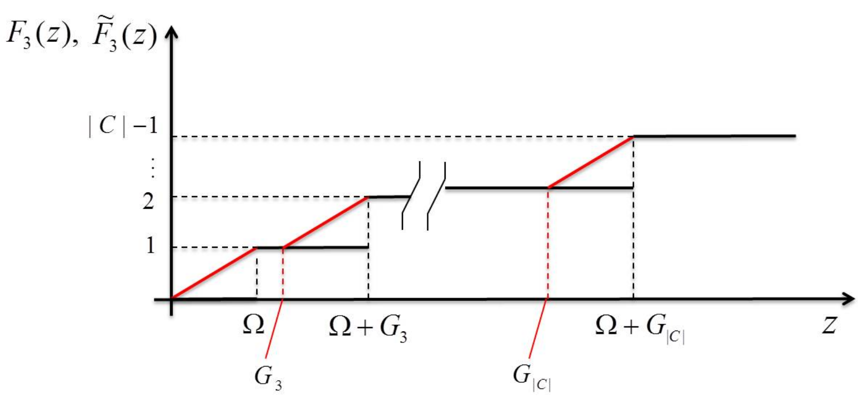

Figure 4: Lifting functions $F_3(z)$ and $\tilde{F}_3(z)$ on $\mathbb{R}_+$, where $\tilde{F}_3(z)$ is marked in red in intervals $[G_{j+1}, \Omega + G_{j+1})$ for $j = 1, \ldots, |C| - 1$

In order to obtain valid lifting coefficients, we construct a piecewise linear overestimation for the approximate lifting function $\tilde{F}_3(z)$. If there exist pieces of the overestimation touching the actual lifting function $F_3(z)$ at two points, we can then obtain the corresponding lifting coefficients by the intersect and slope of the piece. A similar result as Lemma 6 for the approximate lifting function $\tilde{F}_3(z)$ is shown as follows.

Lemma 7 *For each $i \in \mathcal{I} \setminus C$, define set $\Pi_i = \{G_j : \underline{Q}_i \leq \Omega + G_j \leq \bar{Q}_i, j = 2, \ldots, |C|\} \cup \{\underline{Q}_i, \bar{Q}_i\}$, and let $H_i^1 < H_i^2 < \cdots < H_i^{|\Pi_i|}$ be its increasing sorted order. Then the piecewise linear function*

$$\Psi_i(z) = \min_{k=1,\ldots,|\Pi_i|-1} \{\psi_i^k(z)\}, \tag{34}$$

where $\psi_i^k(z) = \tilde{F}_3(H_i^k) + \left(\frac{\tilde{F}_3(H_i^{k+1}) - \tilde{F}_3(H_i^k)}{H_i^{k+1} - H_i^k}\right)(z - H_i^k)$ is an overestimation of $\tilde{F}_3(z)$ on $[\underline{Q}_i, \bar{Q}_i]$, and each $\psi_i^k(z)$ touches $F_3(z)$ at two points $(H_i^k, F_3(H_i^k))$ and $(H_i^{k+1}, F_3(H_i^{k+1}))$ with possible exception when $k = 1$ or $|\Pi_i| - 1$.

Proof: The detailed proof is shown in Appendix C.9. ■

With the piecewise linear overestimation of the approximate lifting function $\tilde{F}_3(z)$ defined in Lemma 7, the cover inequality (21) can be lifted as follows.

Proposition 13 *For a given cover seed inequality* (21)*, the lifted inequality*

$$\sum_{i \in C} y_i + \sum_{i \in \mathcal{I} \setminus C} (\alpha_i y_i + \beta_i x_i) \geq |C| - 1 \tag{35}$$

is valid for conv(Z), where (α_i, β_i) is in the form of

$$\left(\frac{\tilde{F}_3(H_i^k) H_i^{k+1} - \tilde{F}_3(H_i^{k+1}) H_i^k}{H_i^{k+1} - H_i^k}, \quad \frac{\tilde{F}_3(H_i^{k+1}) - \tilde{F}_3(H_i^k)}{H_i^{k+1} - H_i^k}\right) \tag{36}$$

corresponding to each $k = 1, \ldots, |\Pi_i| - 1$. Furthermore, inequality (35) *is facet-defining for conv(Z) if $|\Pi_i| \geq 4$ and $k = 2, \ldots, |\Pi_i| - 2$ for each $i \in \mathcal{I} \setminus C$.*

Proof: The detailed proof is shown Appendix C.10. ■

Separation 3: For a given point $(\hat{y}, \hat{x}) \in \mathbb{B}^I \times \mathbb{R}_+^I$, similar to Separations 1 and 2, we use a greedy approach to find an initial cover C such that $D < \sum_{i=1}^{|C|-1} \bar{Q}_{[i]}$. Note here that the requirement for a cover is equivalent to $\sum_{i=3}^{|C|} \bar{Q}_{[i]} < D < \sum_{i=1}^{|C|-1} \bar{Q}_{[i]}$ when the conditions $R = |C| - 1$ and $D < \sum_{i=1}^{|C|-1} \bar{Q}_{[i]}$ as described in Lemma 4 are enforced. Now, we check if $\sum_{i=3}^{|C|} \bar{Q}_{[i]} < D$ is satisfied. If not, then we stop with no inequality found; if yes, then we let $\Omega = D - \sum_{i=3}^{|C|} \bar{Q}_{[i]}$ and generate lifting coefficients (α_i, β_i) for each $i \in \mathcal{I} \setminus C$ according to (36). Here, we use a similar approach as described in Separation 2 to find the corresponding (α_i, β_i) for each $i \in \mathcal{I} \setminus C$. Finally, we check if the lifted cover inequality (35) is violated: if not, then we stop with no inequality found; if yes, then we return the corresponding inequality (35), which cuts off the given point $(\hat{y}, \hat{x})$.

6 Computational Experiments

In this section, we conduct a computational experiment to assess the effectiveness of our cutting planes, by implementing a branch-and-cut algorithm and testing it on an illustrative test instance, under various parameter settings. In the following, we describe the test instances in Section 6.1 and report the computational results in Section 6.2.

6.1 Test Instances

Our computational experiment is based on a power system data set containing 118 buses, 33 generators, and 186 transmission lines (available online at `http://www.ee.washington.edu/research/pstca/pf118/pg_tca118bus.htm`). In this experiment, we create four groups of instances by considering subsets of the generators with cardinalities 15, 20, 25, and 30, respectively. For each group of instances, we consider two types of scenario tree structures, including (1) a binary tree, i.e., each non-leaf node in the scenario tree has two children, and (2) a ternary tree, i.e., each non-leaf node in the scenario tree has three children. In addition, we consider instances with 9 and 10 time periods for the binary tree structure, and instances with 6 and 7 time periods for the ternary tree structure. We use *a*-*b*-*c* to denote an instance, with a denoting the number of generators, b denoting the scenario tree structure, and c denoting the number of time periods. For example, an instance 20-2-10 considers 20 generators on a binary tree with the operational time horizon to be 10 time periods.

For the UC characteristics of the generators in this experiment, we set the minimum up and down time of each generator to be between 1 and 10 periods, the start-up and shut-down ramp rate limits of each generator to be 50% of the upper generation limit, i.e., $\bar{V}^b = \bar{B}^b = 0.5\bar{Q}_b$,

$\forall b \in \mathcal{I}, \forall n \in \mathcal{V}$, and the ramp-up and ramp-down rate limits of each generator to be 30% of the upper generation limit, i.e., $V^b = B^b = 0.3\bar{Q}_b$, $\forall b \in \mathcal{I}, \forall n \in \mathcal{V}$. In addition, the load uncertainty is assumed to be uniform within the interval $[0.9\bar{N}, 1.1\bar{N}]$, where $\bar{N}$ represents the nominal load. As stated in Section 2, we assume no curtailment of renewable energy, combine the uncertainties in the generation and load sides in this experiment, and regard the combined uncertainty as the net load uncertainty.

6.2 Computational Results

The algorithms were implemented using CPLEX 12.1 callable library, and all the numerical experiments were conducted at an Intel Quad Core 2.40GHz with 2GB memory. The computational results of this experiment are reported in Tables 1 and 2, which collect the root node and branch-and-cut algorithm statistics respectively. In both tables, we let `Top` represent the results obtained by the algorithm if we consider only the turn on/off inequalities, `Top+R` represent the results obtained by considering both turn on/off and ramping inequalities, and `Top+R+C` represent the results obtained by considering turn on/off, ramping, and lifted cover inequalities.

In Table 1, we compare the root node gaps obtained by using different approaches. The column OriGap reports the root node gap of the original linear programming relaxation of the MSUC formulation as compared to the best upper bound obtained by the default CPLEX on the corresponding instance. Similarly, the columns `Top`, `Top+R` and `Top+R+C` report the root node gaps obtained by implementing turn on/off inequalities, both turn on/off and ramping inequalities, and all three types of inequalities, respectively. We observe that OriGap is between 1% and 3% in all the instances, and the root node gap is reduced to within 1% after adding the turn on/off inequalities, is further reduced to within 0.5% after adding the ramping inequalities, and is finally reduced to within 0.3% after adding the lifted cover inequalities. This observation implies that all the three groups of cutting planes proposed in this paper can effectively strengthen the linear programming relaxation of the MSUC formulation.

In Table 2, we compare the performance of our branch-and-cut algorithm obtained by implementing different approaches, with a one-hour time limit. In the columns `Top`, `Top+R` and `Top+R+C`, we report the final optimality gap and the CPU time (in seconds), where we use a "$***$" to indicate that the corresponding instance is not solved to the default CPLEX optimality tolerance within

Table 1: Root node optimality gaps

Instance	OriGap (%)	`Top` (%)	`Top+R` (%)	`Top+R+C` (%)
15-2-9	1.883	0.036	0.018	0.016
15-2-10	2.170	0.200	0.105	0.095
15-3-6	2.547	0.020	0.020	0.010
15-3-7	1.968	0.044	0.024	0.020
20-2-9	1.883	0.027	0.017	0.017
20-2-10	2.191	0.216	0.112	0.111
20-3-6	2.549	0.006	0.006	0.006
20-3-7	1.969	0.065	0.044	0.044
25-2-9	2.109	0.346	0.157	0.127
25-2-10	2.478	0.559	0.291	0.177
25-3-6	2.558	0.061	0.044	0.044
25-3-7	2.003	0.048	0.043	0.025
30-2-9	1.895	0.053	0.029	0.023
30-2-10	2.214	0.833	0.315	0.257
30-3-6	2.612	0.059	0.054	0.054
30-3-7	2.003	0.026	0.026	0.025

3600s. Note that in each instance, the final optimality gap reported is obtained by comparing the best lower bound to the best upper bound obtained within the test, rather than the best upper bound obtained for the instance throughout all the tests. That is, the experiment for testing each family of inequalities (e.g., turn on/off inequalities, ramping inequalities, or lifted cover inequalities) is self-contained in terms of obtaining lower and upper bounds. Hence, the final optimality gaps reported in Table 2 can be viewed as a conservative estimate of the true optimality gap. From results shown in Table 2, we first observe that the proposed algorithm can solve the comparatively easy instances to optimality within the one-hour time limit (see, e.g., instances 15-3-6 by all variants and instances 20-2-9 by the variant `Top+R+C`). Further, in such instances the solution time in the variants `Top+R` and `Top+R+C` are significantly shorter than those of `Top` variant in most instances. Second, we observe that in all the instances where optimality cannot be achieved, the proposed algorithm solves them to within 1% optimality. For the comparatively harder instances (e.g., instances 25-2-10 and 30-2-10), we observe that the optimality gaps are significantly reduced by adding the ramping inequalities, and are further reduced to within 0.2% by adding the lifted cover inequalities. Both of the above observations indicate that the proposed branch-and-cut algorithm can effectively solve our MSUC test instances and obtain (near) optimal solutions. In particular, by implementing the cutting planes generated by the ramping inequalities and lifted cover inequalities, we can solve the easier instances to optimality in shorter times and solve the harder instances to within smaller optimality gaps, than by executing branch-and-cut without these specialized cutting planes.

Table 2: Gap and run-time statistics for branch-and-cut algorithms

Instance	Top		Top+R		Top+R+C	
	Gap (%)	CPU secs	Gap (%)	CPU secs	Gap (%)	CPU secs
15-2-9	0.00	2277	0.00	1336	0.00	1332
15-2-10	0.10	***	0.08	***	0.05	***
15-3-6	0.00	86	0.00	54	0.00	52
15-3-7	0.02	***	0.02	***	0.01	***
20-2-9	0.01	***	0.01	***	0.00	3445
20-2-10	0.12	***	0.10	***	0.10	***
20-3-6	0.00	231	0.00	223	0.00	212
20-3-7	0.03	***	0.02	***	0.02	***
25-2-9	0.13	***	0.09	***	0.09	***
25-2-10	0.43	***	0.27	***	0.16	***
25-3-6	0.00	910	0.00	909	0.00	693
25-3-7	0.03	***	0.03	***	0.02	***
30-2-9	0.02	***	0.02	***	0.01	***
30-2-10	0.74	***	0.27	***	0.10	***
30-3-6	0.00	1723	0.00	1319	0.00	1299
30-3-7	0.01	***	0.01	***	0.01	***

7 Conclusions

In this paper, we introduced efficient cutting planes for MSUC and developed a branch-and-cut algorithm to solve the problem. In particular, we discovered several families of strong valid inequalities for the minimum up/down time polytope, the ramping polytope, and the economic dispatch polytope, respectively. In our approach, by exploring the problem structures, we first obtained a convex hull representation of the minimum up/down time polytope under the multi-stage stochastic scenario tree setting. Then, by taking advantage of the minimum up/down time restrictions, we discovered the sequence independent lifting properties for the original ramping constraints, which lead to obtaining stronger valid ramping inequalities. The derived two family of inequalities are facet-defining under mild conditions and more importantly, the numbers for these inequalities are polynomial functions of the input size of the problem. Finally, corresponding to each node in the scenario tree, by analyzing the economic dispatch polytope structure, we derived sequence independent and subadditive approximation lifting properties to obtain strong valid inequalities. The separations for these inequalities are NP-hard in general, and we provided efficient heuristic separation algorithms. Due to the same structure of the economic dispatch polytope at each node in the stochastic scenario tree, our lifted cover inequalities can be generated once based on different right hand side values and applied to each node in the scenario tree.

In general, as shown one of the premier methods to improve linear programming relaxation bounds of mixed-integer linear programs, the cutting plane approach helps reduce the optimality gap and ultimately speed up the corresponding branch-and-cut algorithm to solve mixed-integer linear programs. In our computational experiment, the derived strong valid inequalities for the three polytopes helped significantly strengthen the linear programming relaxation of the MSUC problem. The computational experiment results based on a power system under various data parameter settings verified the empirical effectiveness of the proposed cutting planes and the corresponding branch-and-cut algorithm. The proposed approach is very promising and in future research, we will explore how to integrate the cutting plane approach with the progressive hedging heuristic and other decomposition algorithms to solve large-scale problems.

Acknowledgments

The authors would like to thank Ming Zhao at SAS and Xing Wang at Alstom Grid, for insight discussions. This research was supported in part by the U.S. National Science Foundation under CAREER Award CMMI-0942156 and in part from the Office of Advanced Scientific Computing Research within the Department of Energy's Office of Science. Sandia National Laboratories is a multi-program laboratory managed and operated by Sandia Corporation, a wholly owned subsidiary of Lockheed Martin Corporation, for the U.S. Department of Energy's National Nuclear Security Administration under contract DE-AC04-94AL85000.

References

[1] S. Ahmed, M. Tawarmalani, and N. V. Sahinidis. A finite branch and bound algorithm for two-stage stochastic integer programs. *Mathematical Programming*, 100(2):355–377, 2004.

[2] A. Botterud, J. Wang, V. Miranda, and R. Bessa. Wind power forecasting in U.S. electricity markets. *The Electricity Journal*, 23(3):71–82, 2010.

[3] C. C. Carøe. *Decomposition in stochastic integer programming.* PhD thesis, University of Copenhagen, 1998.

[4] C. C. Carøe and R. Schultz. Dual decomposition in stochastic integer programming. *Operations Research Letters*, 24(1–2):37–45, 1999.

[5] S. Cerisola, Á. Baíllo, J. M. Fernández-López, A. Ramos, and R. Gollmer. Stochastic power generation unit commitment in electricity markets: A novel formulation and a comparison of solution methods. *Operations Research*, 57(1):32–46, 2009.

[6] K. Chung, J.-P. Richard, and M. Tawarmalani. Lifted inequalities for 0-1 mixed-integer bilinear covering sets. *Preprint available at: www.optimization-online.org*, 2011.

[7] A. Faruqui and S. Sergici. Household response to dynamic pricing of electricity: a survey of 15 experiments. *Journal of Regulatory Economics*, 38(2):193–225, 2010.

[8] S. Fink, C. Mudd, K. Porter, and B. Morgenstern. Wind energy curtailment case studies. In *National Renewable Energy Laboratory Report*, 2009. Available online at http://www.nrel.gov/docs/fy10osti/46716.pdf.

[9] Z. Gu, G. Nemhauser, and M. Savelsbergh. Sequence independent lifting in mixed integer programming. *Journal of Combinatorial Optimization*, 4(1):109–129, 2000.

[10] X. Guan, P. B. Luh, J. A. Yan, and J. A. Amalfi. An optimization-based method for unit commitment. *International Journal of Electrical Power & Energy Systems*, 14(1):9–17, 1992.

[11] Y. Guan, S. Ahmed, and G. Nemhauser. Cutting planes for multistage stochastic integer programs. *Operations Research*, 57(2):287–298, 2009.

[12] Y. Guan, S. Ahmed, G. Nemhauser, and A. J. Miller. A branch-and-cut algorithm for the stochastic uncapacitated lot-sizing problem. *Mathematical Programming*, 105(1):55–84, 2006.

[13] K. Huang and S. Ahmed. The value of multi-stage stochastic programming in capacity planning under uncertainty. *Operations Research*, 57(4):893–904, 2009.

[14] D. Klabjan, G. Nemhauser, and C. Tovey. The complexity of cover inequality separation. *Operations Research Letters*, 23(1–2):35–40, 1998.

[15] W. K. Klein Haneveld, L. Stougie, and M. H. van der Vlerk. An algorithm for the construction of convex hulls in simple integer recourse programming. *Annals of Operations Research*, 64(1):67–81, 1996.

[16] G. Laporte and F. V. Louveaux. The integer L-shaped method for stochastic integer programs with complete recourse. *Operations Research Letters*, 13:133–142, 1993.

[17] A. Løkketangen and D. L. Woodruff. Progressive hedging and tabu search applied to mixed integer (0, 1) multi-stage stochastic programming. *Journal of Heuristics*, 2(2):111–128, 1996.

[18] G. Lulli and S. Sen. A branch-and-price algorithm for multi-stage stochastic integer programming with application to stochastic batch-sizing problems. *Management Science*, 50(6):786–796, 2004.

[19] P. Meibom, R. Barth, H. Brand, and C. Weber. Wind power integration studies using a multi-stage stochastic electricity system model. In *IEEE Power Engineering Society General Meeting, 2007.*, pages 1–4, 2007.

[20] G. Nemhauser and L. A. Wolsey. *Integer and Combinatorial Optimization.* Wiley, 1988.

[21] A. Papavasiliou, S. Oren, and R.P. O'Neill. Reserve requirements for wind power integration: A scenario-based stochastic programming framework. *IEEE Transactions on Power Systems*, 26(4):2197–2206, 2011.

[22] D. Rajan and S. Takriti. Minimum up/down polytopes of the unit commitment problem with start-up costs. *IBM Research Report*, 2005.

[23] J.-P. Richard and M. Tawarmalani. Lifting inequalities: a framework for generating strong cuts for nonlinear programs. *Mathematical Programming*, 121(1):61–104, 2010.

[24] S. Sen and J. L. Higle. The C^3 Theorem and a D^2 algorithm for large scale stochastic mixed-integer programming: Set convexification. *Mathematical Programming*, 104(1):1–20, 2005.

[25] G. Sheble and G. Fahd. Unit commitment literature synopsis. *IEEE Transactions on Power Systems*, 9(1):128–135, 1994.

[26] S. Takriti, J. R. Birge, and E. Long. A stochastic model for the unit commitment problem. *IEEE Transactions on Power Systems*, 11(3):1497–1508, 1996.

[27] C. Tseng, S. Oren, C. Cheng, C. Li, A. Svoboda, and R. Johnson. A transmission-constrained unit commitment method in power system scheduling. *Decision Support Systems*, 24(3):297–310, 1999.

[28] J. Wang, A. Botterud, V. Miranda, C. Monteiro, and G. Sheble. Impact of wind power forecasting on unit commitment and dispatch. In *8th Int. Workshop on Large-Scale Integration of Wind Power into Power Systems, Bremen, Germany*, Oct. 2009.

[29] J. Wang, M. Shahidehpour, and Z. Li. Security-constrained unit commitment with volatile wind power generation. *IEEE Transactions on Power Systems*, 23(3):1319–1327, 2008.

[30] Q. Wang, Y. Guan, and J. Wang. A chance-constrained two-stage stochastic program for unit commitment with uncertain wind power output. *IEEE Transactions on Power Systems*, 27(1):206–215, 2012.

[31] S. Wang, S. Shahidehpour, D. Kirschen, S. Mokhtari, and G. Irisarri. Short-term generation scheduling with transmission and environmental constraints using an augmented Lagrangian relaxation. *IEEE Transactions on Power Systems*, 10(3):1294–1301, 1995.

[32] B. Whittle, J. Yu, S. Teng, and J. Mickey. Reliability unit commitment in ERCOT. In *Proceedings of the IEEE Power Engineering Society General Meeting*, Montreal, Que., 2006.

[33] L. Wolsey. Valid inequalities and superadditivity for 0-1 integer programs. *Mathematics of Operations Research*, 2(1):66–77, 1977.

[34] L. Wu, M. Shahidehpour, and T. Li. Stochastic security-constrained unit commitment. *IEEE Transactions on Power Systems*, 22(2):800–811, 2007.

Appendix A Proofs for the Minimum Up/Down Time Polytope

In this section, we provide detailed proofs for the claims in Section 3. For self-containedness, we recall that the minimum up/down time polytope is the convex hull of P, where

$$P := \Big\{(y,u) \in \mathbb{B}^{|\mathcal{V}|} \times \mathbb{B}^{|\mathcal{V}|-1} : \; y_n - y_{n^-} \le y_i, \quad \forall n \in \mathcal{V} \setminus \{1\}, \forall i \in \mathcal{H}_{\bar{L}}(n), \tag{37}$$

$$y_{n^-} - y_n \le 1 - y_i, \quad \forall n \in \mathcal{V} \setminus \{1\}, \forall i \in \mathcal{H}_{\underline{L}}(n), \tag{38}$$

$$y_n - y_{n^-} \le u_n \le \min\{y_n,\ 1 - y_{n^-}\}, \quad \forall n \in \mathcal{V} \setminus \{1\}\Big\}.$$

Besides, we recall that the turn on/off inequalities on the scenario tree as follows:

$$\text{Turn on inequality:} \qquad \sum_{i=0}^{\bar{L}-1} u_{n_i^-} \le y_n, \qquad \forall n, \text{ such that } t(n) = \bar{L}+1, \ldots, T, \tag{39}$$

$$\text{Turn off inequality:} \qquad \sum_{i=0}^{\underline{L}-1} u_{n_i^-} \le 1 - y_{n_{\underline{L}}^-}, \qquad \forall n, \text{ such that } t(n) = \underline{L}+1, \ldots, T. \tag{40}$$

In the following, we show that the turn on/off inequalities are (i) valid for conv(P), and (ii) sufficient to describe conv(P) together with some trivial inequalities.

Observation 1 *The turn on/off inequalities* (39) *and* (40) *are valid for conv*(P)*.*

Proof: In the following, we show that the turn on/off inequalities are valid for P, which implies that they are valid for conv(P). For the validity of the turn on inequalities, we consider the following cases:

Case 1. Suppose $y_n = 1$. Since $n_0^-, n_1^-, \ldots, n_{\bar{L}-1}^-$ are $\bar{L}$ consecutive nodes on $\mathcal{P}(n)$, we know that $\sum_{i=0}^{\bar{L}-1} u_{n_i^-} \le 1$ because the minimum up time is $\bar{L}$, and so we cannot start up the generator twice within $\bar{L}$ periods.

Case 2. Suppose $y_n = 0$, i.e., the generator is off at node n. In this case, we cannot start up the generator at node n_i^- for any $i = 0, \ldots, \bar{L}-1$, because otherwise y_n would be forced to 1 by the minimum up time constraint (37) in view that $n \in \mathcal{H}_{\bar{L}}(n_i^-)$.

Similarly, we prove the validity of the turn off inequalities by considering the following two cases:

Case 1. Suppose $y_{n_{\underline{L}}^-} = 0$ and assume that $\sum_{i=0}^{\underline{L}-1} u_{n_i^-} \ge 2$ for a contradiction. Since we start up the generator at least twice within $\underline{L}$ periods, we have to shut it down in some period between the two consecutive starting up operations, which violates the minimum down time constraints. Hence, we have $\sum_{i=0}^{\underline{L}-1} u_{n_i^-} \leq 1$.

Case 2. Suppose $y_{n_{\underline{L}}^-} = 1$. By following a similar argument as described in Case 1, it is easy to verify that we cannot start a generator up in the next $\underline{L}$ periods, i.e., $\sum_{i=0}^{\underline{L}-1} u_{n_i^-} \leq 0$. ∎

Next we show that the above proposed inequalities are sufficient to describe conv(P). To prove this, we construct the polytope

$$
\begin{aligned}
Q = \quad \Big\{(y,u) \in \mathbb{R}_+^{|\mathcal{V}|} \times \mathbb{R}_+^{|\mathcal{V}|-1} : \quad & \sum_{i=0}^{\bar{L}-1} u_{n_i^-} \leq y_n, \forall n, \text{ such that } t(n) = \bar{L}+1,\ldots,T, \\
& \sum_{i=0}^{\underline{L}-1} u_{n_i^-} \leq 1 - y_{n_{\underline{L}}^-}, \forall n, \text{ such that } t(n) = \underline{L}+1,\ldots,T, \\
& y_n - y_{n^-} \leq u_n, \forall n \in \mathcal{V} \setminus \{1\}\Big\},
\end{aligned}
$$

and show that $Q = \text{conv}(P)$. Since Q consists of valid inequalities for conv(P), we have conv$(P) \subseteq Q$. To prove the reserve part, i.e., $Q \subseteq \text{conv}(P)$, our plan is to prove that (i) all the extreme points of Q are integral, and (ii) all the integral points of Q are contained in P. We summarize these two steps as Claims 1 and 2 as follows. The readers can find a similar proof for the deterministic case in [22]. In the following, we let functions $y_n(\cdot)$ and $u_n(\cdot)$ to reflect the corresponding component of vector $z = (\bar{y}, \bar{u}) \in Q$, i.e., $y_n(z) = \bar{y}_n$ for each $n \in \mathcal{V}$ and $u_n(z) = \bar{u}_n$ for each $n \in \mathcal{V} \setminus \{1\}$. Accordingly, we define function $v_n(z) = y_{n^-}(z) - y_n(z) + u_n(z)$ for each $z \in Q$ and each $n \in \mathcal{V} \setminus \{1\}$.

Claim 1 *Let $z = (\bar{y}, \bar{u}) \in Q$. There exists a set of integral vectors $(z^s)_{s \in S}$ in Q, where S is an index set, such that:*

(i) $z = \sum_{s \in S} \lambda_s z^s$ for some $\lambda_s \in \mathbb{R}_+$ and $\sum_{s \in S} \lambda_s = 1$.

(ii) For each node $n \in \mathcal{V} \setminus \{1\}$, let $S_n^u = \{s \in S : u_n(z^s) = 1\}$. Then we have $\bar{u}_n = \sum_{s \in S_n^u} \lambda_s$.

(iii) For each node $n \in \mathcal{V} \setminus \{1\}$, let $S_n^d = \{s \in S : v_n(z^s) = 1\}$. Then we have $\bar{v}_n = \bar{y}_{n^-} - \bar{y}_n + \bar{u}_n = \sum_{s \in S_n^d} \lambda_s$.

Proof: Note here that we employ $\bar{v}_n$ and $v_n(z^s)$ defined above as auxiliary variables. For $n \in [2, |\mathcal{V}|]_{\mathbb{Z}}$, we let Q_n be the projection of Q onto the $(y_1, \ldots, y_n, u_2, \ldots, u_n)$ space, i.e., $Q_n = \text{proj}_{\mathbb{R}^{n \times (n-1)}} Q$.

For example, $Q_{|\mathcal{V}|} = Q$. Accordingly, we let $z_n = (\bar{y}_1, \ldots, \bar{y}_n, \bar{u}_2, \ldots, \bar{u}_n)$ be the projection of z on $\mathbb{R}^{n\times(n-1)}$, such that $z_n \in Q_n$. We prove claims (i), (ii), and (iii) by induction on n.

Base Case: For $n = 2$, $z_n = (\bar{y}_1, \bar{y}_2, \bar{u}_2)$, we pick $z^1, \ldots, z^4 \in Q_n$, such that

$$
\begin{aligned}
&y_1(z^1) = 1,\ y_2(z^1) = 0,\ u_2(z^1) = 0,\\
&y_1(z^2) = 0,\ y_2(z^2) = 0,\ u_2(z^2) = 0,\\
&y_1(z^3) = 0,\ y_2(z^3) = 1,\ u_2(z^3) = 1,\ \text{and}\\
&y_1(z^4) = 1,\ y_2(z^4) = 1,\ u_2(z^4) = 0,
\end{aligned}
$$

and thus $z_n = (\bar{y}_1 - \bar{y}_2 + \bar{u}_2)z^1 + (1 - \bar{y}_1 - \bar{u}_2)z^2 + \bar{u}_2 z^3 + (\bar{y}_2 - \bar{u}_2)z^4$. In other words, we have $z_n = \sum_{s=1}^4 \lambda_s z^s$ with $\lambda_1 = \bar{y}_1 - \bar{y}_2 + \bar{u}_2$, $\lambda_2 = 1 - \bar{y}_1 - \bar{u}_2$, $\lambda_3 = \bar{u}_2$, and $\lambda_4 = \bar{y}_2 - \bar{u}_2$. Clearly, $\lambda_1, \ldots, \lambda_4 \geq 0$ and $\sum_{s=1}^4 \lambda_s = 1$. In this case, we have $S_2^u = \{3\}$ and $S_2^d = \{1\}$, and we can verify that $\bar{u}_2 = \sum_{s\in S_2^u} \lambda_s = \lambda_3$ and $\bar{v}_2 = \sum_{s\in S_2^d} \lambda_s = \lambda_1$. Hence, claims (i), (ii), and (iii) hold for Q_2.

Induction Case: Assume that claims (i), (ii), and (iii) hold for Q_n for each $n \in [2, |\mathcal{V}|-1]_{\mathbb{Z}}$, where we define $[a, b]_{\mathbb{Z}}$ to be the set $\{a, a+1, \ldots, b\}$ for integers a and b. By assumption, there exists a set S' and $(w^s)_{s\in S'}$ in Q_n associated with $(\mu_s)_{s\in S'}$, such that $z_n = \sum_{s\in S'} \mu_s w^s$, $\mu_s \geq 0$ for each $s \in S'$, and $\sum_{s\in S'} \mu_s = 1$. Besides, we have $\bar{u}_m = \sum_{s\in (S')_m^u} \mu_s$ and $\bar{v}_m = \sum_{s\in (S')_m^d} \mu_s$ for each $m \in [2, n]_{\mathbb{Z}}$.

Now, we show the claims hold for Q_{n+1}, i.e., there exists an index set S and $(z^s)_{s\in S}$ in Q_{n+1} associated with $(\lambda_s)_{s\in S}$, such that $z_{n+1} = \sum_{s\in S} \lambda_s z^s$, $\lambda_s \geq 0$ for each $s \in S$, and $\sum_{s\in S} \lambda_s = 1$. Besides, we show that $\bar{u}_m = \sum_{s\in S_m^u} \lambda_s$ and $\bar{v}_m = \sum_{s\in S_m^d} \lambda_s$ for each $m \in [2, n+1]_{\mathbb{Z}}$. We are going to obtain $(z^s)_{s\in S}$ by augmenting each point w^s in two more dimensions y_{n+1} and u_{n+1} while keeping the components in the other dimensions the same. Accordingly, we are going to divide weight μ_s for the points augmented from w^s. As shown in Figure 5, there are three possible ways of augmenting w^s, for which we make the following observations:

1. We can augment w^s by (a) starting up the generator, (b) continuing keeping the generator on, or (c) continuing keeping the generator off or shutting it down at node $n+1$. We note that the feasibility of z^{s_1}, z^{s_2}, and z^{s_3} (i.e., if they belong to Q_{n+1}) depends on w^s. However, there are at least one of them being feasible. We let $f(s) := \{s_i : z^{s_i} \in Q_{n+1}, i = 1, 2, 3\}$ represent the set of the indices of feasible augmented vectors. After augmenting w^s for each $s \in S'$, we let $S = \cup_{s\in S'} f(s)$ and collect all the feasible augmented vectors $(z^s)_{s\in S}$, where $z^s \in Q_{n+1}$ for each

$s \in S$.

2. Since we inherit components $y_1, \ldots, y_n$ and $u_1, \ldots, u_n$ from w^s and let $\lambda_{s_1} + \lambda_{s_2} + \lambda_{s_3} = \mu_s$ for each $s \in S'$, it follows that however we assign the weights λ_{s_1}, λ_{s_2}, and λ_{s_3}, claims (i), (ii), and (iii) are automatically satisfied except in the dimensions y_{n+1} and u_{n+1}.

$$\begin{tikzcd}[column sep=small]
 & & & y_{n+1} \arrow[d] & u_{n+1} \arrow[d] \\
 & & (\lambda_{s_1}) \quad z^{s_1} = (y_1, \ldots, y_n, & 1, u_2, \ldots, u_n, & 1) \\
(\mu_s) \quad w^s = (y_1, \ldots, y_n, u_2, \ldots, u_n) \arrow[rru] \arrow[rr] \arrow[rrd] & & (\lambda_{s_2}) \quad z^{s_2} = (y_1, \ldots, y_n, & 1, u_2, \ldots, u_n, & 0) \\
 & & (\lambda_{s_3}) \quad z^{s_3} = (y_1, \ldots, y_n, & 0, u_2, \ldots, u_n, & 0)
\end{tikzcd}$$

$$(\mu_s = \lambda_{s_1} + \lambda_{s_2} + \lambda_{s_3})$$

Figure 5: Augmenting the points in induction step

Next we show how to assign the weights λ_{s_1}, λ_{s_2}, and λ_{s_3} for each w^s, $s \in S'$, to make the claims valid for dimensions y_{n+1} and u_{n+1}.

To satisfy claim (ii), we assign a weight to λ_{s_1} (in which case $u_{n+1}(z^{s_1}) = 1$) for sufficiently many vectors w^s such that $\bar{u}_{n+1} = \sum_{s \in S^u_{n+1}} \lambda_s$. Since we can divide and assign the weight μ_s continuously, we only have to show there is sufficient amount of weight μ_s we can assign from. For each $s \in S'$, to assign weight μ_s to λ_{s_1} for vector w^s, we augment w^s to z^{s_1} by starting up the generator at node $n+1$. We observe that $z^{s_1} \in Q_{n+1}$ and accordingly μ_s can be assigned from for each $s \in S'$ except the following two cases. First, $z^{s_1} \notin Q_{n+1}$ if the generator is on at node $(n+1)^-$, i.e., $z^{s_1} \notin Q_{n+1}$ if $s \in S'_1 = \{s \in S' : y_{(n+1)^-}(w^s) = 1\}$. Second, in view of the minimum down time constraints, $z^{s_1} \notin Q_{n+1}$ if $s \notin S'_1$ but the generator is shut down at any nodes among $(n+1)^-_1, \ldots, (n+1)^-_{\underline{L}-1}$, i.e., $z^{s_1} \notin Q_{n+1}$ if $s \in (S')^d_{(n+1)^-_i}$ for any $i \in [1, \underline{L}-1]_{\mathbb{Z}}$. Hence, the total amount of μ_s we can assign weight from is $1 - \sum_{s \in S_1'} \mu_s - \sum_{i=1}^{\underline{L}-1} \sum_{s \in (S')^d_{(n+1)_i^-}} \mu_s$. It follows that

$$
\begin{aligned}
1 - \sum_{s \in S_1'} \mu_s - \sum_{i=1}^{\underline{L}-1} \sum_{s \in (S')^d_{(n+1)_i^-}} \mu_s &= 1 - \bar{y}_{(n+1)^-} - \sum_{i=1}^{\underline{L}-1} \bar{v}_{(n+1)_i^-} \\
&= 1 - \bar{y}_{(n+1)^-} - \sum_{i=1}^{\underline{L}-1} \left(\bar{y}_{(n+1)_{i+1}^-} - \bar{y}_{(n+1)_i^-} + \bar{u}_{(n+1)_i^-} \right) \\
&= 1 - \bar{y}_{(n+1)^-} + \bar{y}_{(n+1)^-} - \bar{y}_{(n+1)_{\underline{L}}^-} - \sum_{i=1}^{\underline{L}-1} \bar{u}_{(n+1)_i^-} \\
&\geq 1 - \bar{y}_{(n+1)_{\underline{L}}^-} - \left(1 - \bar{y}_{(n+1)_{\underline{L}}^-} - \bar{u}_{n+1} \right) \; = \; \bar{u}_{n+1},
\end{aligned}
$$

where the first equality is due to the definitions of S_1' and $(S')^d_{(n+1)_i^-}$, the second equality follows from $\bar{v} = y_{n^-} - y_n + u_n$ for each $n \in \mathcal{V}$, and the last inequality follows from the turn off inequality. Hence, we have sufficient amount of μ_s to assign from, and accordingly claim (ii) is satisfied.

To satisfy claim (iii), we assign a weight to λ_{s_3} for sufficiently many vectors w^s such that $\bar{v}_{n+1} = \sum_{s \in S^d_{n+1}} \lambda_s$. Similarly, for claim (iii), we only have to check if we have sufficient amount of μ_s to assign from. For each $s \in S'$, to assign weight μ_s to λ_{s_3}, we augment w^s to z^{s_3} by shutting down the generator at node $n+1$. First, we observe that $z^{s_3} \in Q_{n+1}$ only if $s \in S_1'$. Second, in view of the minimum up time constraint, $z^{s_3} \notin Q_{n+1}$ if $s \in S_1'$ but the generator is started up at any nodes among $(n+1)_1^-, \ldots, (n+1)_{\bar{L}-1}^-$ i.e., $z^{s_3} \notin Q_{n+1}$ if $s \in (S')^u_{(n+1)_i^-}$ for any $i \in [1, \bar{L}-1]_{\mathbb{Z}}$. Hence, the total amount of μ_s we can assign weight from is $\sum_{s \in S_1'} \mu_s - \sum_{i=1}^{\bar{L}-1} \sum_{s \in (S')^u_{(n+1)_i^-}} \mu_s$. It follows that

$$
\sum_{s \in S_1'} \mu_s - \sum_{i=1}^{\bar{L}-1} \sum_{s \in (S')^u_{(n+1)_i^-}} \mu_s \; = \; \bar{y}_{(n+1)^-} - \sum_{i=1}^{\bar{L}-1} \bar{u}_{(n+1)_i^-} \; \geq \; \bar{y}_{(n+1)^-} + (\bar{u}_{n+1} - \bar{y}_{n+1}) \; = \; \bar{v}_{n+1},
$$

where the first equality follows from the definitions of S_1' and $(S')^u_{(n+1)_i^-}$, and the inequality is due to the turn on inequality. Hence, we have sufficient amount of μ_s to assign from, and accordingly claim (iii) is satisfied.

Finally, we show that claim (i) is satisfied. Since claims (ii) and (iii) are satisfied, there exists $(\lambda_s)_{s \in S}$ such that $\bar{u}_{n+1} = \sum_{s \in S^u_{n+1}} \lambda_s$ and $\bar{v}_{n+1} = \sum_{s \in S^d_{n+1}} \lambda_s$. For claim (i), we show $\bar{u}_{n+1} = \sum_{s \in S} \lambda_s u_{n+1}(z^s)$ and $\bar{y}_{n+1} = \sum_{s \in S} \lambda_s y_{n+1}(z^s)$. To that end, we have

$$
\bar{u}_{n+1} = \sum_{s \in S^u_{n+1}} \lambda_s \; = \sum_{s \in S : u_{n+1}(z^s) = 1} \lambda_s \; = \; \sum_{s \in S} \lambda_s u_{n+1}(z^s).
$$

Furthermore, we have

$$\begin{aligned}
\sum_{s\in S}\lambda_s y_{n+1}(z^s) &= \sum_{s\in S: y_{n+1}(z^s)=1}\lambda_s \\
&= \sum_{s\in S: y_{n+1}(z^s)=1, y_{(n+1)^-}(z^s)=0}\lambda_s + \sum_{s\in S: y_{n+1}(z^s)=1, y_{(n+1)^-}(z^s)=1}\lambda_s \\
&= \sum_{s\in S^u_{n+1}}\lambda_s + \sum_{s\in S: y_{(n+1)^-}(z^s)=1}\lambda_s - \sum_{s\in S: y_{n+1}(z^s)=0, y_{(n+1)^-}(z^s)=1}\lambda_s \\
&= \bar{u}_{n+1} + \sum_{s\in S'_1}\mu_s - \sum_{s\in S^d_{n+1}}\lambda_s \\
&= \bar{u}_{n+1} + \bar{y}_{(n+1)^-} - \bar{v}_{n+1} \\
&= \bar{y}_{n+1}.
\end{aligned}$$

To sum up, we have proved that claims (i), (ii), and (iii) also hold for Q_{n+1}, and hence the desired conclusions follow by induction. ∎

Claim 2 *Any integral vectors in Q belongs to P, i.e., $Q\cap\mathbb{B}^{|\mathcal{V}|\times(|\mathcal{V}|-1)}\subseteq P$.*

Proof: For any given integral vector $(\bar{y},\bar{u})\in Q\cap\mathbb{B}^{|\mathcal{V}|\times(|\mathcal{V}|-1)}$, we show that $(\bar{y},\bar{u})$ satisfies both minimum up and down time constraints.

To see $(\bar{y},\bar{u})$ satisfying the minimum up time constraint (37), we pick any nodes $n\in\mathcal{V}\setminus\{1\}$ and $i\in\mathcal{H}_{\bar{L}}(n)$. We only discuss the case when $\bar{y}_{n^-}=0$ and $\bar{y}_n=1$, since constraint (37) is clearly satisfied in all the other cases. We have $\bar{u}_n=1$ because $\bar{u}_n\geq\bar{y}_n-\bar{y}_{n^-}=1$, and it follows that $\sum_{j=0}^{\bar{L}-1}\bar{u}_{i_j^-}=1$ since $i\in\mathcal{H}_{\bar{L}}(n)$. Hence, we have $\bar{y}_i=1$ by the turn on inequality, and so constraint (37) is satisfied.

To see $(\bar{y},\bar{u})$ satisfying the minimum down time constraint (38), we pick any nodes $n\in\mathcal{V}\setminus\{1\}$ and $i\in\mathcal{H}_{\underline{\mathrm{L}}}(n)$. We only discuss the case when $\bar{y}_{n^-}=1$ and $\bar{y}_n=0$, since constraint (38) is clearly satisfied in all the other cases. We have $\bar{v}_n=1$ because $\bar{v}_n=\bar{u}_n-\bar{y}_n+\bar{y}_{n^-}\geq 1$, and it follows that $\sum_{j=0}^{\underline{\mathrm{L}}-1}\bar{v}_{i_j^-}=1$ since $i\in\mathcal{H}_{\underline{\mathrm{L}}}(n)$. Hence, we have $\bar{y}_i\leq 1-\sum_{j=0}^{\underline{\mathrm{L}}-1}\bar{v}_{i_j^-}=0$ by the turn off inequality, and so constraint (38) is satisfied. ∎

The above two claims immediately give us the main conclusion on the minimum up/down time polytope.

Theorem 1 *$Q = conv(P)$.*

Proof: Since Q consists of valid inequalities for $\text{conv}(P)$, we have $\text{conv}(P) \subseteq Q$.

On the other side, by the conclusion (i) in Claim 1 and Claim 2, we know that for any $z \in Q$, there exists vectors $(z^s)_{s\in S}$ in P, where S is an index set, such that $z = \sum_{s\in S} \lambda_s z^s$ for some $\lambda_s \in \mathbb{R}_+$ and $\sum_{s\in S} \lambda_s = 1$. It follows that $Q \subseteq \text{conv}(P)$. ■

Appendix B Proofs for the Ramping Polytope

In this section, we provide detailed proofs of the claims for the ramping polytope in Section 4.

B.1 Proof of Proposition 1

Proof: We prove that $\dim\left(\text{conv}(Y_n^i)\right) = 3T - 1$ by generating $3T$ affinely independent points in $\text{conv}(Y_n^i)$. Since $0 \in \text{conv}(Y_n^i)$, we generate another $3T-1$ linearly independent points in $\text{conv}(Y_n^i)$. For notation brevity, we let $[a, b]_{\mathbb{Z}}$ represent $[a, b] \cap \mathbb{Z}$, i.e., $\{a, a+1, \ldots b\}$ for integers a and b. First, we have $(\bar{y}^s, \bar{u}^s, \bar{x}^s) \in \text{conv}(Y_n^i)$ for each $s \in [1, T]_{\mathbb{Z}}$, where

$$\bar{y}_r^s = \begin{cases} 1, & \text{if } r \in [1, s]_{\mathbb{Z}}, \\ 0, & \text{o.w.} \end{cases}, \quad \bar{u}^s = 0, \text{ and } \bar{x}_r^s = \begin{cases} \bar{B}, & \text{if } r \in [1, s], \\ 0, & \text{o.w.} \end{cases}.$$

Second, we have $(\hat{y}^s, \hat{u}^s, \hat{x}^s) \in \text{conv}(Y_n^i)$ for each $s \in [2, T]_{\mathbb{Z}}$, where

$$\hat{y}_r^s = \begin{cases} 1, & \text{if } r \in [s, T]_{\mathbb{Z}}, \\ 0, & \text{o.w.} \end{cases}, \quad \hat{u}_r^s = \begin{cases} 1, & \text{if } r = s, \\ 0, & \text{o.w.} \end{cases}, \text{ and } \hat{x}_r^s = \begin{cases} \underline{Q}, & \text{if } r \in [s, T]_{\mathbb{Z}}, \\ 0, & \text{o.w.} \end{cases}.$$

Third, we have $(\tilde{y}^s, \tilde{u}^s, \tilde{x}^s) \in \text{conv}(Y_n^i)$ for each $s \in [1, T]_{\mathbb{Z}}$, where

$$\tilde{y}_r^s = 1, \ \forall r \in [1, T]_{\mathbb{Z}}, \quad \tilde{u}^s = 0, \text{ and } \tilde{x}_r^s = \begin{cases} \underline{Q}, & \text{if } r = s, \\ \bar{Q}, & \text{o.w.} \end{cases}.$$

Note here that the T points $(\tilde{y}^s, \tilde{u}^s, \tilde{x}^s)_{s=1}^T$ are different since $\bar{Q} > \underline{Q}$ by observation (6). Besides, it is clear that the points $(\bar{y}^s, \bar{u}^s, \bar{x}^s)_{s=1}^T$, $(\hat{y}^s, \hat{u}^s, \hat{x}^s)_{s=2}^T$ and $(\tilde{y}^s, \tilde{u}^s, \tilde{x}^s)_{s=1}^T$ are linearly independent, which completes the proof. ■

B.2 Proof of Proposition 4

Proof: **(Validity)** We discuss the following four cases based on the values of y_{t-1} and y_t.

(i) If $y_{t-1} = y_t = 0$, inequality (10) is satisfied since $x_{t-1} = 0$ and $x_t = 0$ due to constraint constraint (1f) in the definition of Y_n^i in (5).

(ii) If $y_{t-1} = 0$ and $y_t = 1$, inequality (10) is satisfied since $x_{t-1} = 0$ due to constraint (1f), and $x_t \geq \underline{Q} \geq \min\{\underline{Q}, \bar{B} - B\}$ where the first inequality is due to constraint (1f) in the definition of Y_n^i in (5).

(iii) If $y_{t-1} = 1$ and $y_t = 0$, inequality (10) reduces to ramping-down constraint (1i) in the definition of Y_n^i in (5).

(iv) If $y_{t-1} = y_t = 1$, inequality (10) is satisfied since $x_{t-1} - x_t \leq B \leq \max\{\bar{B} - \underline{Q}, B\} = \bar{B} - \min\{\underline{Q}, \bar{B} - B\}$, where the first inequality is due to ramping-down constraint (1i).

(Facet-defining) To prove that inequality (10) is facet-defining for $\mathrm{conv}(Y_n^i)$, we generate $3T - 1$ affinely independent points in $\mathrm{conv}(Y_n^i)$ that satisfy inequality (10) at equality. Since $0 \in \mathrm{conv}(Y_n^i)$ and satisfies inequality (10) at equality, we generate another $3T - 2$ linearly independent points as shown in Table 3. We categorize these points in five groups as follows.

(i) For each $s \in [1, t-1]_{\mathbb{Z}}$, $(\bar{y}^s, \bar{u}^s, \bar{x}^s) \in \mathrm{conv}(Y_n^i)$ with

$$\bar{y}_r^s = \begin{cases} 1, & \text{if } r \in [1, s]_{\mathbb{Z}}, \\ 0, & \text{o.w.} \end{cases}, \quad \bar{u}^s = 0, \text{ and } \bar{x}_r^s = \begin{cases} \bar{B}, & \text{if } r \in [1, s]_{\mathbb{Z}}, \\ 0, & \text{o.w.} \end{cases}.$$

Meanwhile, $(\bar{y}^s, \bar{u}^s, \bar{x}^s)$ satisfies inequality (10) at equality because $\bar{y}_t^s = \bar{x}_t^s = 0$ and $\bar{x}_{t-1}^s = \bar{B}\bar{y}_{t-1}^s$.

(ii) For each $s \in [t, T]_{\mathbb{Z}}$, $(\bar{y}^s, \bar{u}^s, \bar{x}^s) \in \mathrm{conv}(Y_n^i)$ with

$$\bar{y}_r^s = \begin{cases} 1, & \text{if } r \in [1, s]_{\mathbb{Z}}, \\ 0, & \text{o.w.} \end{cases}, \quad \bar{u}^s = 0, \text{ and } \bar{x}_r^s = \begin{cases} \bar{B} + B, & \text{if } r[1, s]_{\mathbb{Z}} \setminus \{t\}, \\ \bar{B}, & \text{if } r = t, \\ 0, & \text{o.w.} \end{cases}.$$

Note here that $\bar{B} + B < \bar{Q}$ by one condition in Proposition 4. Meanwhile, $(\bar{y}^s, \bar{u}^s, \bar{x}^s)$ satisfies inequality (10) at equality because $\bar{x}_{t-1}^s - \bar{x}_t^s - \bar{B}\bar{y}_{t-1}^s + \min\{\underline{Q}, \bar{B} - B\}\bar{y}_t^s = (\bar{B} + B) - \bar{B} - \bar{B} + (\bar{B} - B) = 0$, where the first equality is due to $\bar{y}_{t-1}^s = \bar{y}_t^s = 1$ and the condition $\underline{Q} = \bar{B} - B$.

(iii) For each $s \in [2, t-1]_{\mathbb{Z}}$, $(\tilde{y}^s, \tilde{u}^s, \tilde{x}^s) \in \mathrm{conv}(Y_n^i)$ with

$$\tilde{y}_r^s = \begin{cases} 1, & \text{if } r \in [s, T]_{\mathbb{Z}}, \\ 0, & \text{o.w.} \end{cases}, \quad \tilde{u}_r^s = \begin{cases} 1, & \text{if } r = s, \\ 0, & \text{o.w.} \end{cases}, \text{ and } \tilde{x}_r^s = \begin{cases} \bar{V}, & \text{if } r \in [s, t-1]_{\mathbb{Z}}, \\ \bar{V} - B, & \text{if } r \in [t, T]_{\mathbb{Z}}, \\ 0, & \text{o.w.} \end{cases}.$$

Note here that $\bar{V} - B \geq \bar{B} - B = \underline{\mathrm{Q}}$, where the inequality is due to the condition $\bar{V} \geq \bar{B}$. Meanwhile, $(\tilde{y}^s, \tilde{u}^s, \tilde{x}^s)$ satisfies inequality (10) at equality because $\tilde{x}_{t-1}^s - \tilde{x}_t^s - \bar{B}\tilde{y}_{t-1}^s + \min\{\underline{\mathrm{Q}}, \bar{B} - B\}\tilde{y}_t^s = \bar{V} - (\bar{V} - B) - \bar{B} + (\bar{B} - B) = 0$, where the first equality uses the condition $\underline{\mathrm{Q}} = \bar{B} - B$.

(iv) For each $s \in [t, T]_{\mathbb{Z}}$, $(\tilde{y}^s, \tilde{u}^s, \tilde{x}^s) \in \mathrm{conv}(Y_n^i)$ with

$$\tilde{y}_r^s = \begin{cases} 1, & \text{if } r \in [s, T]_{\mathbb{Z}}, \\ 0, & \text{o.w.} \end{cases}, \quad \tilde{u}_r^s = \begin{cases} 1, & \text{if } r = s, \\ 0, & \text{o.w.} \end{cases}, \text{ and } \tilde{x}_r^s = \begin{cases} \underline{\mathrm{Q}}, & \text{if } r \in [s, T]_{\mathbb{Z}}, \\ 0, & \text{o.w.} \end{cases}.$$

Meanwhile, $(\tilde{y}^s, \tilde{u}^s, \tilde{x}^s)$ satisfies inequality (10) at equality because $\tilde{y}_{t-1}^s = \tilde{x}_{t-1}^s = 0$ and $\tilde{x}_t^s = \underline{\mathrm{Q}}\tilde{y}_t^s$.

(v) For each $s \in [0, T]_{\mathbb{Z}} \setminus \{t-1, t\}$, $(\dot{y}^s, \dot{u}^s, \dot{x}^s) \in \mathrm{conv}(Y_n^i)$ with

$$\dot{y}_r^s = 1, \forall r \in [1, T]_{\mathbb{Z}}, \quad \dot{u}^s = 0, \text{ and } \dot{x}_r^s = \begin{cases} \bar{Q} - B, & \text{if } r = s \text{ or } r = t, \\ \bar{Q}, & \text{o.w.} \end{cases}.$$

Note here that when $s = 0$, $\dot{x}_r^s = \bar{Q}$ for each $r \in [1, T]_{\mathbb{Z}}$ except that $\dot{x}_t^s = \bar{Q} - B$. Meanwhile, $(\dot{y}^s, \dot{u}^s, \dot{x}^s)$ satisfies inequality (10) at equality because $\dot{x}_{t-1}^s - \dot{x}_t^s - \bar{B}\dot{y}_{t-1}^s + \min\{\underline{\mathrm{Q}}, \bar{B} - B\}\dot{y}_t^s = \bar{Q} - (\bar{Q} - B) - \bar{B} + (\bar{B} - B) = 0$, where the first equality uses the condition $\underline{\mathrm{Q}} = \bar{B} - B$.

Finally, these five groups of points are clearly linearly independent from Table 3. Hence, we have generated $(t-1) + (T-t+1) + (t-2) + (T-t+1) + (T-1) = 3T-2$ linearly independent points as desired, and accordingly inequality (10) is facet-defining for $\mathrm{conv}(Y_n^i)$. ■

B.3 Proof of Lemma 1

Proof: **(Validity)** Since the variable $u_{t+i} = 0$ for each $i \in W$ where $W = [K - \bar{L} + 1, \min\{T - t, \underline{\mathrm{L}}\}]_{\mathbb{Z}}$, we have $u_{t+i} = 0$ for each $i \in [0, K]_{\mathbb{Z}}$ because (i) $K - \bar{L} + 1 \leq 0$ by the definition of K, and (ii) $K \leq \min\{T - t, \underline{\mathrm{L}}\}$ by $t \leq T - K$ and the definition of K. It follows that sequence $\{y_{t+i}, i \in [0, K]_{\mathbb{Z}}\}$

Group	y				u			x					
	1	2	$\cdots t-1$	$t \cdots T$	2	$\cdots t-1$	$t \cdots T$	1	2	$\cdots\ t-1$	t	$\cdots$	T
1	1	0	$\cdots\ 0$	$0 \cdots 0$	0	$\cdots\ 0$	$0 \cdots 0$	$\bar{B}$	0	$\cdots\ 0$	0	$\cdots$	0
	1	1	$\cdots\ 0$	$0 \cdots 0$	0	$\cdots\ 0$	$0 \cdots 0$	$\bar{B}$	$\bar{B}$	$\cdots\ 0$	0	$\cdots$	0
	$\vdots$	$\vdots$	$\vdots$	$\vdots\ \ \vdots$	$\vdots$	$\vdots$	$\vdots\ \ \vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
	1	1	$\cdots\ 1$	$0 \cdots 0$	0	$\cdots\ 0$	$0 \cdots 0$	$\bar{B}$	$\bar{B}$	$\cdots\ \bar{B}$	0	$\cdots$	0
2	1	1	$\cdots\ 1$	$1 \cdots 0$	0	$\cdots\ 0$	$0 \cdots 0$	$\bar{B}+B$	$\bar{B}+B$	$\cdots \bar{B}+B$	$\bar{B}$	$\cdots$	0
	$\vdots$	$\vdots$	$\vdots$	$\vdots\ \ \vdots$	$\vdots$	$\vdots$	$\vdots\ \ \vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
	1	1	$\cdots\ 1$	$1 \cdots 1$	0	$\cdots\ 0$	$0 \cdots 0$	$\bar{B}+B$	$\bar{B}+B$	$\cdots \bar{B}+B$	$\bar{B}$	$\cdots$	$\bar{B}+B$
3	0	1	$\cdots\ 1$	$1 \cdots 1$	1	$\cdots\ 0$	$0 \cdots 0$	0	$\bar{V}$	$\cdots\ \bar{V}$	$\bar{V}-\bar{B}$	$\cdots$	$\bar{V}-\bar{B}$
	$\vdots$	$\vdots$	$\vdots$	$\vdots\ \ \vdots$	$\vdots$	$\vdots$	$\vdots\ \ \vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
	0	0	$\cdots\ 1$	$1 \cdots 1$	0	$\cdots\ 1$	$0 \cdots 0$	0	0	$\cdots\ \bar{V}$	$\bar{V}-\bar{B}$	$\cdots$	$\bar{V}-\bar{B}$
4	0	0	$\cdots\ 0$	$1 \cdots 1$	0	$\cdots\ 0$	$1 \cdots 0$	0	0	$\cdots\ 0$	$\underline{Q}$	$\cdots$	$\underline{Q}$
	$\vdots$	$\vdots$	$\vdots$	$\vdots\ \ \vdots$	$\vdots$	$\vdots$	$\vdots\ \ \vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
	0	0	$\cdots\ 0$	$0 \cdots 1$	0	$\cdots\ 0$	$0 \cdots 1$	0	0	$\cdots\ 0$	0	$\cdots$	$\underline{Q}$
5	1	1	$\cdots\ 1$	$1 \cdots 1$	0	$\cdots\ 0$	$0 \cdots 0$	$\bar{Q}$	$\bar{Q}$	$\cdots\ \bar{Q}$	$\bar{Q}-B$	$\cdots$	$\bar{Q}$
	1	1	$\cdots\ 1$	$1 \cdots 1$	0	$\cdots\ 0$	$0 \cdots 0$	$\bar{Q}-B$	$\bar{Q}$	$\cdots\ \bar{Q}$	$\bar{Q}-B$	$\cdots$	$\bar{Q}$
	$\vdots$	$\vdots$	$\vdots$	$\vdots\ \ \vdots$	$\vdots$	$\vdots$	$\vdots\ \ \vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
	1	1	$\cdots\ 1$	$1 \cdots 1$	0	$\cdots\ 0$	$0 \cdots 0$	$\bar{Q}$	$\bar{Q}$	$\cdots\ \bar{Q}$	$\bar{Q}-B$	$\cdots$	$\bar{Q}-B$

Table 3: $3T-2$ linearly independent points for inequality (10)

is nonincreasing. That is, for some $i \in [0, K]_{\mathbb{Z}}$, $y_{t+i} = 0$ implies $y_{t+i'} = 0$ for each $i' \in [i, K]_{\mathbb{Z}}$, and $y_{t+i} = 1$ implies $y_{t+i'} = 1$ for each $i' \in [0, i]_{\mathbb{Z}}$. We prove the validity of inequality (14) by discussing the following cases based on the value of $\{y_{t+i}, i \in [0, K]_{\mathbb{Z}}\}$.

1. If $y_{t+i} = 0$ for each $i \in [0, K]_{\mathbb{Z}}$, then inequality (14) is clearly satisfied.

2. If $y_{t+i} = 1$, $y_{t+i+1} = 0$ for some $i \in [0, K-1]_{\mathbb{Z}}$, then from the nonincreasing property of sequence $\{y_{t+i}, i \in [0, K]_{\mathbb{Z}}\}$ we have $y_{t+i'} = 1$ for each $i' \in [0, i]_{\mathbb{Z}}$ and $y_{t+i'} = 0$ for each $i \in [i+1, K]_{\mathbb{Z}}$. It follows that $x_t \leq \bar{B} + (i-1)B$ by ramping-down constraint (1i) in the definition of Y_n^i in (5). Meanwhile, since the right-hand-side (RHS) of inequality (14) is $\bar{B}y_t + \sum_{i=1}^{K-1} By_{t+i} + (\bar{Q} - \bar{B} - (K-1)^+B)y_{t+K} = \bar{B} + (i-1)B$, inequality (14) is satisfied.

3. If $y_{t+i} = 1$ for each $i \in [0, K]_{\mathbb{Z}}$, then inequality (14) reduces to $x_t \leq \bar{Q}$, which is clearly satisfied.

(Facet-defining) Since we fix $u_{t+i} = 0$ for each $i \in W$ in the ramping polytope $\text{conv}(Y_n^i)$, the lower-dimensional space of $\text{conv}(Y_n^i)$ thus obtained has dimension $(3T-1) - |W| = 2T - 1 + (t + K - \bar{L}) + (T - t - \underline{L})^+$. Since $0 \in \text{conv}(Y_n^i)$, we prove the facet-defining property by generating $3T - |W| - 2$

linearly independent points $\text{conv}(Y_n^i)$ satisfying inequality (14) at equality. We categorize the points into four groups as follows.

1. For each $s \in [1, T-1]_{\mathbb{Z}}$, $(\bar{y}^s, \bar{u}^s, \bar{x}^s) \in \text{conv}(Y_n^i)$ with

$$\bar{y}_r^s = \begin{cases} 1, & \text{if } r \in [1, s]_{\mathbb{Z}}, \\ 0, & \text{o.w.} \end{cases}, \quad \bar{u}^s = 0, \text{ and } \bar{x}_r^s = \begin{cases} \min\{\bar{Q}, \bar{B} + (s-r)B\}, & \text{if } r \in [1, s]_{\mathbb{Z}}, \\ 0, & \text{o.w.} \end{cases},$$

because (i) $\min\{\bar{Q}, \bar{B} + (s-r)B\} \leq \bar{Q}$ for each $r \in [1, s]_{\mathbb{Z}}$, and (ii) $\bar{x}_r^s$ is nonincreasing and $\bar{x}_r^s - \bar{x}_{r+1}^s \leq B$ for each $r \in [1, s-1]_{\mathbb{Z}}$. Meanwhile, inequality (14) is satisfied at equality because:

   (a) If $s \in [0, t-1]_{\mathbb{Z}}$, then inequality (14) is clearly satisfied at equality since $\bar{x}_t^s = 0$ and $\bar{y}_{t+i}^s = 0$ for each $i \in [0, K]_{\mathbb{Z}}$.

   (b) If $s \in [t, t+K-1]_{\mathbb{Z}}$, then $\bar{x}_t^s = \bar{B} + (s-t)B$ since $\bar{B} + (s-t)B \leq \bar{B} + (K-1)B < \bar{Q}$ due to $K - 1 = K_2$, $\bar{y}_i^s = 1$ for each $i \in [t, s]_{\mathbb{Z}}$, and $\bar{y}_i^s = 0$ for each $i \in [s+1, t+K]$. It follows that the RHS of inequality (14) is $\bar{B} y_t + \sum_{i=1}^{K-1} B y_{t+i} + (\bar{Q} - \bar{B} - (K-1)^+ B) y_{t+K} = \bar{B} + (s-t)B$, which implies that inequality (14) is satisfied at equality.

   (c) If $s \in [t+K, T-1]_{\mathbb{Z}}$, then $\bar{x}_t^s = \bar{Q}$ since $\bar{B} + (s-t)B \geq \bar{B} + (K_2+1)B \geq \bar{Q}$ due to $K = K_2 + 1$, and $\bar{y}_{t+i}^s = 1$ for each $i \in [0, K]_{\mathbb{Z}}$. It follows that the RHS of inequality (14) is $\bar{Q}$, which implies that inequality (14) is satisfied at equality.

   For $s = T$, $(\bar{y}^s, \bar{u}^s, \bar{x}^s) \in \text{conv}(Y_n^i)$ with

$$\bar{y}_r^s = 1, \ \forall r \in [1, T]_{\mathbb{Z}}, \quad \bar{u}^s = 0, \text{ and } \bar{x}_r^s = \bar{Q}, \ \forall r \in [1, T]_{\mathbb{Z}}.$$

   Meanwhile, inequality (14) is satisfied at equality because $\bar{x}_t^s = \bar{Q}$ and RHS of inequality (14) is $\bar{Q}$ as well in view that $\bar{y}_{t+i}^s = 1$ for each $i \in [0, K]_{\mathbb{Z}}$.

2. For each $s \in [2, t+K-\bar{L}]_{\mathbb{Z}}$, $(\tilde{y}^s, \tilde{u}^s, \tilde{x}^s) \in \text{conv}(Y_n^i)$ with

$$\tilde{y}_r^s = \begin{cases} 1, & \text{if } r \in [s, t+K-1]_{\mathbb{Z}}, \\ 0, & \text{o.w.} \end{cases}, \quad \tilde{u}_r^s = \begin{cases} 1, & \text{if } r = s, \\ 0, & \text{o.w.} \end{cases},$$

$$\text{and } \tilde{x}_r^s = \begin{cases} \min\{\bar{V} + (r-s)V, \bar{B} + K_2 B\}, & \text{if } r \in [s, t-1]_{\mathbb{Z}}, \\ \bar{B} + (K_2 - r + t)B, & \text{if } r \in [t, t+K-1]_{\mathbb{Z}}, \\ 0, & \text{o.w.} \end{cases}$$

because (i) $\tilde{x}_r^s \leq \bar{B} + K_2 B < \bar{Q}$ for each $r \in [1, T]_{\mathbb{Z}}$, (ii) $\tilde{x}_r^s$ is nondecreasing on set $[s, t-1]_{\mathbb{Z}}$ as r increases and $\tilde{x}_{r+1}^s - \tilde{x}_r^s \leq V$ for each $r \in [s, t-2]_{\mathbb{Z}}$, (iii) $\tilde{x}_r^s$ is nonincreasing on set $[t, t+K]_{\mathbb{Z}}$ as r increases and $\tilde{x}_r^s - \tilde{x}_{r+1}^s = B$ for each $r \in [t, t+K-1]_{\mathbb{Z}}$, and (iv) $\tilde{x}_t^s \geq \tilde{x}_{t-1}^s$ and $\tilde{x}_t^s - \tilde{x}_{t-1}^s \leq V$, since if $\tilde{x}_{t-1}^s = \bar{B} + K_2 B$ we have $\tilde{x}_t^s - \tilde{x}_{t-1}^s = 0$, and if $\tilde{x}_{t-1}^s = \bar{V} + (t-s-1)V$ we have

$$
\begin{aligned}
\tilde{x}_t^s - \tilde{x}_{t-1}^s &= \bar{B} + K_2 B - \bar{V} + (s-t+1)V \\
&\leq \bar{B} + K_2 B - \bar{V} + (K - \bar{L} + 1)V \\
&= \bar{B} + K_2 B - (\bar{V} + (\bar{L} - K)V) + V \\
&\leq V,
\end{aligned}
$$

where the first inequality is due to $s \leq t + K - \bar{L}$, and the last inequality is due to the condition $\bar{V} + (\bar{L} - K)V \geq \bar{B} + K_2 B$. Meanwhile, inequality (14) is satisfied at equality. Indeed, since $\tilde{y}_{t+i} = 1$ for each $i \in [0, K-1]_{\mathbb{Z}}$ and $\tilde{y}_{t+K} = 0$, we have RHS of inequality (14) is $\bar{B} + K_2 B$, which agrees with $\tilde{x}_t^s = \bar{B} + K_2 B$.

3. For $s \in [t + \underline{L} + 1, T]_{\mathbb{Z}}$, $(\dot{y}^s, \dot{u}^s, \dot{x}^s) \in \operatorname{conv}(Y_n^i)$ with

$$
\dot{y}_r^s = \begin{cases} 1, & \text{if } r \in [s, T]_{\mathbb{Z}}, \\ 0, & \text{o.w.} \end{cases}, \quad \dot{u}_r^s = \begin{cases} 1, & \text{if } r = s, \\ 0, & \text{o.w.} \end{cases}, \text{ and } \dot{x}_r^s = \begin{cases} \underline{Q}, & \text{if } r \in [s, T]_{\mathbb{Z}}, \\ 0, & \text{o.w.} \end{cases}.
$$

Meanwhile, inequality (14) is satisfied at equality because $\dot{x}_t^s = 0$, and $\dot{y}_{t+i}^s = 0$ for each $i \in [0, K]_{\mathbb{Z}}$ since $t + K < t + \underline{L} + 1 \leq s$ due to the definition of K and $s \geq t + \underline{L} + 1$.

4. For $s \in [1, T]_{\mathbb{Z}} \setminus \{t\}$, $(\ddot{y}^s, \ddot{u}^s, \ddot{x}^s) \in \operatorname{conv}(Y_n^i)$ with

$$
\ddot{y}_r^s = 1, \forall r \in [1, T]_{\mathbb{Z}}, \quad \ddot{u}^s = 0, \text{ and } \ddot{x}_r^s = \begin{cases} \bar{Q} - \epsilon, & \text{if } r = s, \\ \bar{Q}, & \text{o.w.} \end{cases},
$$

where $\epsilon := \min\{\bar{Q} - \underline{Q}, V, B\}$. Note that both ramping constraints (1h) and (1i) are satisfied since the generator is always on and the difference of generation quantities between two consecutive time periods is at most ϵ. Meanwhile, inequality (14) is satisfied at equality because $\ddot{x}_t^s = \bar{Q}$, and $\ddot{y}_{t+i}^s = 1$ for each $i \in [0, K]_{\mathbb{Z}}$.

Finally, these five groups of points are linearly independent because variables $\bar{y}^s$ for $s \in [1, T]_{\mathbb{Z}}$ consist of a lower triangle matrix with all nonzero components being one, variables $\tilde{u}^s$ for $s \in [2, t + K - \bar{L}]_{\mathbb{Z}}$ and $\dot{u}^s$ for $s \in [t + \underline{L} + 1, T]_{\mathbb{Z}}$ consist of a matrix with each row and column having a component one and any other component zero, and variables $\ddot{x}^s$ for $s \in [1,T]_{\mathbb{Z}} \setminus \{t\}$ consist of a matrix with each row and column having a component $\bar{Q} - \epsilon$ and any other component $\bar{Q}$. Therefore, we have generated $T + (t + K - \bar{L} - 1) + (T - t - \underline{L})^+ + (T - 1) = 2T - 2 + (t + K - \bar{L}) + (T - t - \underline{L})^+$ linearly independent points as desired, and accordingly the proof is complete. ■

Appendix C Proofs for the Economic Dispatch Polytope

C.1 Proof of Proposition 8

Proof: To show $\dim(\mathrm{conv}(Z)) = 2I$, we find $2I + 1$ affinely independent points in $\mathrm{conv}(Z)$. First, by assumption (18), we have $(\bar{x}^j, \bar{y}^j) \in \mathrm{conv}(Z)$ for each $j \in \mathcal{I}$ with

$$\bar{y}_i^j = \begin{cases} 0, & \text{if } i = j, \\ 1, & \text{o.w.} \end{cases}, \quad \bar{x}_i^j = \begin{cases} 0, & \text{if } i = j, \\ \bar{Q}_i, & \text{o.w.} \end{cases}, \forall i \in \mathcal{I}.$$

Similarly, we have $(\hat{x}^j, \hat{y}^j) \in \mathrm{conv}(Z)$ for each $j \in \mathcal{I}$ with

$$\hat{y}_i^j = 1, \quad \hat{x}_i^j = \begin{cases} \underline{Q}_i, & \text{if } i = j, \\ \bar{Q}_i, & \text{o.w.} \end{cases}, \forall i \in \mathcal{I}.$$

In addition, we have $(\tilde{x}, \tilde{y}) \in \mathrm{conv}(Z)$ with each $\tilde{x}_i = 1$ and each $\tilde{y}_i = \bar{Q}_i$. Finally, it is clear that points $\{(\bar{x}^j, \bar{y}^j), j \in \mathcal{I}\}$, $\{(\hat{x}^j, \hat{y}^j), j \in \mathcal{I}\}$ and $(\tilde{x}, \tilde{y})$ are affinely independent, and so the proof is thus complete. ■

C.2 Proof of the facet-defining property of Lemma 4

Proof: First, by following a similar argument in the proof of Proposition 8, we can show that $\mathrm{conv}(Z_C)$ is full-dimensional, i.e., $\dim\Big(\mathrm{conv}(Z_C)\Big) = 2|C|$. We omit the proof here for notation brevity. Next, to prove the the facet-defining property of inequality (21), we find $2|C|$ affinely independent points in $\mathrm{conv}(Z_C)$ that satisfy (21) at equality. Since $D < \sum_{i=1}^{|C|-1} \bar{Q}_{[i]}$, we have $\sum_{i \in C} \bar{Q}_i - \bar{Q}_j > D$ for each $j \in C$. It follows that $(\bar{y}^j, \bar{x}^j) \in \mathrm{conv}(Z_C)$, where

$$\bar{y}_i^j = \begin{cases} 0, & \text{if } i = j, \\ 1, & \text{o.w.} \end{cases}, \quad \bar{x}_i^j = \begin{cases} 0, & \text{if } i = j, \\ \bar{Q}_i, & \text{o.w.} \end{cases}, \forall i \in C.$$

Since $D < \sum_{i=1}^{|C|-1} \bar{Q}_{[i]}$ we let $\epsilon = \sum_{i=1}^{|C|-1} \bar{Q}_{[i]} - D$ and accordingly $\sum_{i \in C} \bar{Q}_i - \bar{Q}_j - \epsilon \geq D$. We also sort the indices in C in an increasing order and denote this order as permutation σ, i.e., we have $C = \{\sigma_1, \dots, \sigma_{|C|}\}$ and $\sigma_1 < \cdots < \sigma_{|C|}$. It follows that $(\hat{y}^{\sigma_j}, \hat{x}^{\sigma_j}) \in \operatorname{conv}(Z_C)$ for each $j = 1, \dots, |C| - 1$, where

$$\bar{y}_i^{\sigma_j} = \begin{cases} 0, & \text{if } i = j, \\ 1, & \text{o.w.} \end{cases}, \quad \bar{x}_i^{\sigma_j} = \begin{cases} 0, & \text{if } i = j, \\ \bar{Q}_i - \epsilon, & \text{if } i = j+1, \\ \bar{Q}_i, & \text{o.w.} \end{cases}, \forall i \in C.$$

In addition, $(\hat{y}^{\sigma_{|C|}}, \hat{x}^{\sigma_{|C|}}) \in \operatorname{conv}(Z_C)$, where

$$\bar{y}_i^{\sigma_{|C|}} = \begin{cases} 0, & \text{if } i = |C|, \\ 1, & \text{o.w.} \end{cases}, \quad \bar{x}_i^{\sigma_{|C|}} = \begin{cases} 0, & \text{if } i = |C|, \\ \bar{Q}_i - \epsilon, & \text{if } i = 1, \\ \bar{Q}_i, & \text{o.w.} \end{cases}, \forall i \in C.$$

Finally, it is clear that $(\bar{y}^j, \bar{x}^j)$ for each $j \in C$ and $(\hat{y}^{\sigma_j}, \hat{x}^{\sigma_j})$ for each $j = 1, \dots, |C|$ satisfy inequality (21) at equality and they are affinely independent, which completes the proof. ■

C.3 Proof of Theorem 3

Proof: **(Lifting function)** We recall from the lifting procedure described in Section 5.2 that lifting function $F_1(z)$ can be represented by the optimal objective value of the optimization problem

$$F_1(z) = \Delta - \min_{y,x} \Big\{ \sum_{i \in C_2} (x_i - \underline{Q}_i y_i) \Big\} \tag{41a}$$

$$\text{s.t. } \sum_{i \in C} x_i \geq D - z, \tag{41b}$$

$$\underline{Q}_i y_i \leq x_i \leq \bar{Q}_i y_i, \quad \forall i \in C, \tag{41c}$$

$$y_i \in \{0, 1\}, \quad \forall i \in C. \tag{41d}$$

Now we compute $F_1(z)$ by discussing the value of z and solving the corresponding problem (41) in the following cases.

1. If $z \in (-\infty, D - \sum_{i \in C} \bar{Q}_i)$, we have $\sum_{i \in C} \bar{Q}_i < D - z$. It follows that any point (y, x) satisfying constraints (41c)-(41d) violates constraint (41b) since $\sum_{i \in C} x_i \leq \sum_{i \in C} \bar{Q}_i < D - z$, where the first inequality follows from constraint (41c) and $y_i \leq 1$ for each $i \in C$. Therefore, problem (41) is infeasible and its optimal objective value is ∞. Accordingly, $F_1(z) = -\infty$.

2. If $z \in [D - \sum_{i \in C} \bar{Q}_i, \Delta)$, then $D - z > \sum_{i \in C_1} \bar{Q}_i + \sum_{i \in C_2} \underline{Q}_i$ by the definition of Δ. We show that the optimal objective value of problem (41) is $\Delta - z$. First, we follow a similar argument

in the proof of Lemma 2 to show that any given point (y, x) satisfying constraints (41b)-(41d) also satisfies inequality $\sum_{i\in C_2}(x_i - \underline{Q}_i y_i) \geq \Delta - z$, which implies that $\Delta - z$ is a lower bound for the optimal objective value of problem (41). To that end, we observe

$$D - z \leq \sum_{i\in C} x_i = \sum_{i\in C_1} x_i + \sum_{i\in C_2}(x_i - \underline{Q}_i y_i + \underline{Q}_i y_i) \leq \sum_{i\in C_1} \bar{Q}_i + \sum_{i\in C_2}(x_i - \underline{Q}_i y_i) + \sum_{i\in C_2} \underline{Q}_i,$$

where the first inequality is due to constraint (41b), and the second inequality follows from the fact that $x_i \leq \bar{Q}_i$ for each $i \in C_1$ and $y_i \leq 1$ for each $i \in C_2$. Therefore, we have $\sum_{i\in C_2}(x_i - \underline{Q}_i y_i) \geq \Delta - z$ by the definition of Δ. Next we show that the lower bound $\Delta - z$ can be attained by a feasible solution (y^*, x^*) to problem (41). Indeed, $D - z > \sum_{i\in C_1} \bar{Q}_i + \sum_{i\in C_2} \underline{Q}_i$ implies that the solution with $y_i^* = 1$ for each $i \in C$, $x_i^* = \bar{Q}_i$ for each $i \in C_1$ and $x_i^* = \underline{Q}_i$ for each $i \in C_2$ satisfies constraints (41c)-(41d) but not constraint (41b). To satisfy constraint (41b), we increase the value of x_i^*, $i \in C_2$, one by one until $\sum_{i\in C} x_i^* = D - z$, which is well defined because $D - z \leq \sum_{i\in C} \bar{Q}_i$. Finally, $\sum_{i\in C_2}(x_i^* - \underline{Q}_i y_i^*) = \sum_{i\in C} x_i^* - \sum_{i\in C_1} x_i^* - \sum_{i\in C_2} \underline{Q}_i = D - z - \sum_{i\in C_1} \bar{Q}_i - \sum_{i\in C_2} \underline{Q}_i = \Delta - z$. Therefore, the optimal objective value of problem (41) is $\Delta - z$, and accordingly $F_1(z) = z$.

3. If $z \geq \Delta$, we have $\sum_{i\in C_1} \bar{Q}_i + \sum_{i\in C_2} \underline{Q}_i \geq D - z$ by the definition of Δ. It follows that point (y^*, x^*) with $y_i^* = 1$ for each $i \in C$, $x_i^* = \bar{Q}_i$ for each $i \in C_1$, and $x_i^* = \underline{Q}_i$ for each $i \in C_2$ is feasible to problem (41) and has an objective value $\sum_{i\in C_2}(x_i^* - \underline{Q}_i y_i^*) = \sum_{i\in C_2}(\underline{Q}_i - \underline{Q}_i) = 0$. Hence, zero is an upper bound of the optimal objective value of problem (41). Meanwhile, any feasible solution (y, x) to problem (41) satisfies $\sum_{i\in C_2}(x_i - \underline{Q}_i y_i) \geq 0$ due to constraint (41c), which implies that zero is a lower bound of the optimal objective value of problem (41). Therefore, the optimal objective value to problem(41) is zero, and accordingly $F_1(z) = \Delta$.

(Subadditivity) We observe that $0 \leq F_1(z) \leq z$ for all $z \in \mathbb{R}_+$ by its definition. For any given $z_1, z_2 \in \mathbb{R}_+$, we assume without loss of generality that $z_1 \leq z_2$. We show $F_1(z_1+z_2) \leq F_1(z_1)+F_1(z_2)$ by discussing the following tow cases on the values of z_1 and z_2:

1. If $z_1, z_2 \in [0, \Delta)$, then $F_1(z_1 + z_2) \leq z_1 + z_2 = F_1(z_1) + F_1(z_2)$, where the inequality is due to $F_1(z) \leq z$ for all $z \in \mathbb{R}_+$.

2. If $z_2 \geq \Delta$, then $F_1(z_1 + z_2) = \Delta \leq F_1(z_1) + \Delta = F_1(z_1) + F_1(z_2)$ where the inequality is due to

$F_1(z_1) \geq 0$. ■

C.4 Proof of Proposition 9

Proof: **(Validity)** Since the lifting function associated with inequality (19) is subadditive, we can lift the inequality in a sequence independent manner. To prove the validity of the lifted inequality, by the lifting procedure described in Section 5.2, we only need to show that $\alpha_i + \beta_i x_i \geq F_1(x_i)$ for each $i \in \mathcal{I} \setminus C$ over the intervals $x_i \in [\underline{Q}_i, \bar{Q}_i]$. Clearly, from the definition of the lifted coefficients in (29), we know that for each $i \in \mathcal{I} \setminus C$:

1. If $\Delta \leq \underline{Q}_i$, then $(\alpha_i, \beta_i) = (\Delta, 0)$ and accordingly $\alpha_i + \beta_i x_i = \Delta \geq F_1(x_i)$ for each $x_i \in [\underline{Q}_i, \bar{Q}_i]$;

2. If $\underline{Q}_i < \Delta < \bar{Q}_i$ and (α_i, β_i) is chosen to be $(\Delta, 0)$, then $\alpha_i + \beta_i x_i = \Delta \geq F_1(x_i)$ for each $x_i \in [\underline{Q}_i, \bar{Q}_i]$;

3. If $\underline{Q}_i < \Delta < \bar{Q}_i$ and (α_i, β_i) is chosen to be $(0, 1)$, then $\alpha_i + \beta_i x_i = x_i \geq F_1(x_i)$ for each $x_i \in [\underline{Q}_i, \bar{Q}_i]$;

4. If $\Delta \geq \bar{Q}_i$, then $(\alpha_i, \beta_i) = (0, 1)$ and accordingly $\alpha_i + \beta_i x_i = x_i \geq F_1(x_i)$ for each $x_i \in [\underline{Q}_i, \bar{Q}_i]$.

Hence, the validity of inequality (28) follows.

(Facet-defining) To show that facet-defining property of the lifted inequality (28), we prove that the inequality

$$\sum_{i \in C_2} (x_i - \underline{Q}_i y_i) + \Delta y_s \geq \Delta \tag{42}$$

is facet-defining for $\text{conv}(Z_{C \cup \{s\}})$. That is, the seed inequality (19) becomes facet-defining for the low-dimensional polytope after lifting the pair (x_s, y_s) with coefficient $(\Delta, 0)$. To that end, we first observe that $(\Delta, 0)$ is a valid lifting coefficient because, for any $i \in C_1$, $\bar{Q}_s \geq \bar{Q}_i + \Delta > \Delta$ by condition (ii). Hence, the lifted inequality (42) is valid for $\text{conv}(Z_{C \cup \{s\}})$. Second, we can use a similar argument as in the proof of Proposition 8 to prove that $\dim\big(\text{conv}(Z_{C \cup \{s\}})\big) = 2|C| + 2$, i.e., $\text{conv}(Z_{C \cup \{s\}})$ is full-dimensional. We omit the detailed proof by generating affinely independent points for notation brevity, and show that the condition of Proposition 8, namely $\sum_{i \in (C \cup \{s\}) \setminus \{b\}} \bar{Q}_i \geq D$ for each $b \in C \cup \{s\}$ is satisfied as follows.

1. If $b \in C_1$, we have $\sum_{i\in(C\cup\{s\})\setminus\{b\}} \bar{Q}_i \geq \sum_{i\in C_1} \bar{Q}_i + \sum_{i\in C_2} \underline{Q}_i + \bar{Q}_s - \bar{Q}_b = D - \Delta + \bar{Q}_s - \bar{Q}_b \geq D$, where the equality is due to the definition of Δ, and the last inequality follows from condition (ii);

2. If $b \in C_2$, we have

$$\begin{aligned}
&\sum_{i\in(C\cup\{s\})\setminus\{b\}} \bar{Q}_i - \left(\sum_{i\in C_1} \bar{Q}_i + \sum_{i\in C_2} \underline{Q}_i + \bar{Q}_s - \underline{Q}_b\right) \\
=\ & \left(\sum_{i\in C_1} \bar{Q}_i + \sum_{i\in C_2} \bar{Q}_i + \bar{Q}_s - \bar{Q}_b\right) - \left(\sum_{i\in C_1} \bar{Q}_i + \sum_{i\in C_2} \underline{Q}_i + \bar{Q}_s - \underline{Q}_b\right) \\
=\ & \sum_{i\in C_2} (\bar{Q}_i - \underline{Q}_i) - (\bar{Q}_b - \underline{Q}_b) \geq 0,
\end{aligned}$$

   and hence $\sum_{i\in(C\cup\{s\})\setminus\{b\}} \bar{Q}_i \geq \sum_{i\in C_1} \bar{Q}_i + \sum_{i\in C_2} \underline{Q}_i + \bar{Q}_s - \underline{Q}_b = D - \Delta + \bar{Q}_s - \underline{Q}_b \geq D$, where the last inequality follows from condition (ii);

3. If $b = s$, we have $\sum_{i\in(C\cup\{s\})\setminus\{b\}} \bar{Q}_i = \sum_{i\in C} \bar{Q}_i \geq D$ by the condition of Lemma 2.

Third, we show that the face defined by inequality (42) is of dimension $2|C| + 1$ by generating $2|C| + 2$ affinely independent points in $\mathrm{conv}(Z_{C\cup\{s\}})$ satisfying (42) at equality (see the rows of Table 4). In Table 4, we categorize these points into six groups and distinguish them by notation $a, b, \ldots, f$ (see the first column of Table 4). For notation brevity, we index each element in C_1 and C_2 by using subscripts $1, \ldots, |C_1|$ and $1, \ldots, |C_2|$ respectively, and highlight $\bar{Q}_i$ and $\underline{Q}_i$ in subsets C_i, $i = 1, 2$, by using superscript i. For example, $\bar{Q}^1_1$ represents the first generation upper bound in C_1. We define each group of points in the rows of Table 4, and show that they belong to $\mathrm{conv}(Z_{C\cup\{s\}})$ and satisfy inequality (42) at equality as follows.

1. For each $k = 1, \ldots, |C_1|$, $a_k := (y, x) \in \mathrm{conv}(Z_{C\cup\{s\}})$ since $\sum_{i\in C\cup\{s\}} x_i = \sum_{i\in C_1} \bar{Q}^1_i - \bar{Q}^1_k + \sum_{i\in C_2} \underline{Q}^2_i + \bar{Q}_s = \bar{Q}_s - \bar{Q}^1_k - \Delta + D \geq D$, where the second equality is due to the definition of Δ, and the last inequality is due to condition (ii). Moreover, a_k satisfies inequality (42) at equality since $x_i = \underline{Q}_i$ for each $i \in C_2$ and $y_s = 1$.

2. For each $k = 1, \ldots, |C_2|$, $b_k := (y, x) \in \mathrm{conv}(Z_{C\cup\{s\}})$ since $\sum_{i\in C\cup\{s\}} x_i = \sum_{i\in C_1} \bar{Q}^1_i + \sum_{i\in C_2} \underline{Q}^2_i - \underline{Q}^1_k + \bar{Q}_s = \bar{Q}_s - \underline{Q}^2_k - \Delta + D \geq D$, where the last inequality is due to condition (ii). Moreover, b_k satisfies inequality (42) at equality since $x_i = \underline{Q}_i y_i$ for each $i \in C_2$ and $y_s = 1$.

3. $c := (y,x) \in \mathrm{conv}(Z_{C\cup\{s\}})$ since $\sum_{i\in C\cup\{s\}} x_i = \sum_{i\in C_1} \bar{Q}_i^1 + \sum_{i\in C_2} \underline{Q}_i^2 + \max\{\underline{Q}_s, \Delta\} \geq \sum_{i\in C_1} \bar{Q}_i^1 + \sum_{i\in C_2} \underline{Q}_i^2 + \Delta \geq D$, where the last inequality is due to the definition of Δ. Moreover, c satisfies inequality (42) at equality since $x_i = \underline{Q}_i$ for each $i \in C_2$ and $y_s = 1$.

4. For each $k = 1, \ldots, |C_1|$, $d_k := (y,x) \in \mathrm{conv}(Z_{C\cup\{s\}})$ since $\sum_{i\in C\cup\{s\}} x_i = \sum_{i\in C_1} \bar{Q}_i^1 - (\bar{Q}_k^1 - \underline{Q}_k^1) + \sum_{i\in C_2} \underline{Q}_i^2 + \bar{Q}_s \geq \bar{Q}_s - \Delta + D \geq D$, where the last inequality follows from condition (ii). Moreover, d_k satisfies inequality (42) at equality since $x_i = \underline{Q}_i$ for each $i \in C_2$ and $y_s = 1$.

5. For each $k = 1, \ldots, |C_2|$, $e_k := (y,x) \in \mathrm{conv}(Z_{C\cup\{s\}})$ since $\sum_{i\in C\cup\{s\}} x_i = \sum_{i\in C_1} \bar{Q}_i^1 + \sum_{i\in C_2} \underline{Q}_i^2 + \Delta = D$ by the definition of Δ, and $\underline{Q}_i + \Delta \leq \bar{Q}_i$ by condition (i). Moreover, e_k satisfies inequality (42) at equality since $\sum_{i\in C_2}(x_i - \underline{Q}_i) = \Delta$ and $y_s = 0$.

6. $f := (y,x) \in \mathrm{conv}(Z_{C\cup\{s\}})$ since $\sum_{i\in C\cup\{s\}} x_i = \sum_{i\in C_1} \bar{Q}_i^1 + \sum_{i\in C_2} \underline{Q}_i^2 + \bar{Q}_s \geq \sum_{i\in C_1} \bar{Q}_i^1 + \sum_{i\in C_2} \underline{Q}_i^2 + \Delta = D$, where the last inequality is due to condition (ii). Moreover, f satisfies inequality (42) at equality since $x_i = \underline{Q}_i$ for each $i \in C_2$ and $y_s = 1$.

The affine independence is clear from Table 4, and accordingly inequality (42) is facet-defining for $\mathrm{conv}(Z_{C\cup\{s\}})$. Finally, we show the facet-defining property of the lifted inequality (28) by lifting pairs (x_i, y_i) for all $i \in \mathcal{I} \setminus (C \cup \{s\})$ in a way that provides two linearly independent points. To that end, we assume without loss of generality that $\bar{Q}_i > \Delta$ for each $i \in \mathcal{I} \setminus (C \cup \{s\})$, since if $\bar{Q}_i < \Delta$ (since $\bar{Q}_i \neq \Delta$ by condition (iii)) we can insert i into C_1, replace Δ by $\Delta - \bar{Q}_i$, and still keep $\Delta > 0$. Hence, we can lift each pair (x_i, y_i) for all $i \in \mathcal{I} \setminus (C \cup \{s\})$ with lifting coefficient $(\alpha_i, \beta_i) = (\Delta, 0)$, which provides two linearly independent points (Δ, Δ) and $(\bar{Q}_i, \Delta)$ since $\bar{Q}_i > \Delta$. Therefore, the final conclusion follows from Lemma 5. ■

	y									x								
	C_1				C_2				s	C_1				C_2				s
a_1	0	1	$\cdots$	1	1	1	$\cdots$	1	1	0	$\bar{Q}^1_2$	$\cdots$	$\bar{Q}^1_{\lvert C_1\rvert}$	$\underline{Q}^2_1$	$\underline{Q}^2_2$	$\cdots$	$\underline{Q}^2_{\lvert C_2\rvert}$	$\bar{Q}_s$
a_2	1	0	$\cdots$	1	1	1	$\cdots$	1	1	$\bar{Q}^1_1$	0	$\cdots$	$\bar{Q}^1_{\lvert C_1\rvert}$	$\underline{Q}^2_1$	$\underline{Q}^2_2$	$\cdots$	$\underline{Q}^2_{\lvert C_2\rvert}$	$\bar{Q}_s$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$a_{\lvert C_1\rvert}$	1	1	$\cdots$	0	1	1	$\cdots$	1	1	$\bar{Q}^1_1$	$\bar{Q}^1_2$	$\cdots$	0	$\underline{Q}^2_1$	$\underline{Q}^2_2$	$\cdots$	$\underline{Q}^2_{\lvert C_2\rvert}$	$\bar{Q}_s$
b_1	1	1	$\cdots$	1	0	1	$\cdots$	1	1	$\bar{Q}^1_1$	$\bar{Q}^1_2$	$\cdots$	$\bar{Q}^1_{\lvert C_1\rvert}$	0	$\underline{Q}^2_2$	$\cdots$	$\underline{Q}^2_{\lvert C_2\rvert}$	$\bar{Q}_s$
b_2	1	1	$\cdots$	1	1	0	$\cdots$	1	1	$\bar{Q}^1_1$	$\bar{Q}^1_2$	$\cdots$	$\bar{Q}^1_{\lvert C_1\rvert}$	$\underline{Q}^2_1$	0	$\cdots$	$\underline{Q}^2_{\lvert C_2\rvert}$	$\bar{Q}_s$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$b_{\lvert C_2\rvert}$	1	1	$\cdots$	1	1	1	$\cdots$	0	1	$\bar{Q}^1_1$	$\bar{Q}^1_2$	$\cdots$	$\bar{Q}^1_{\lvert C_1\rvert}$	$\underline{Q}^2_1$	$\underline{Q}^2_2$	$\cdots$	0	$\bar{Q}_s$
c	1	1	$\cdots$	1	1	1	$\cdots$	1	1	$\bar{Q}^1_1$	$\bar{Q}^1_2$	$\cdots$	$\bar{Q}^1_{\lvert C_1\rvert}$	$\underline{Q}^2_1$	$\underline{Q}^2_2$	$\cdots$	$\underline{Q}^2_{\lvert C_2\rvert}$	$\max\{\underline{Q}_s, \Delta\}$
d_1	1	1	$\cdots$	1	1	1	$\cdots$	1	1	$\underline{Q}^1_1$	$\bar{Q}^1_2$	$\cdots$	$\bar{Q}^1_{\lvert C_1\rvert}$	$\underline{Q}^2_1$	$\underline{Q}^2_2$	$\cdots$	$\underline{Q}^2_{\lvert C_2\rvert}$	$\bar{Q}_s$
d_2	1	1	$\cdots$	1	1	1	$\cdots$	1	1	$\bar{Q}^1_1$	$\underline{Q}^1_2$	$\cdots$	$\bar{Q}^1_{\lvert C_1\rvert}$	$\underline{Q}^2_1$	$\underline{Q}^2_2$	$\cdots$	$\underline{Q}^2_{\lvert C_2\rvert}$	$\bar{Q}_s$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$d_{\lvert C_1\rvert}$	1	1	$\cdots$	1	1	1	$\cdots$	1	1	$\bar{Q}^1_1$	$\bar{Q}^1_2$	$\cdots$	$\underline{Q}^1_{\lvert C_1\rvert}$	$\underline{Q}^2_1$	$\underline{Q}^2_2$	$\cdots$	$\underline{Q}^2_{\lvert C_2\rvert}$	$\bar{Q}_s$
e_1	1	1	$\cdots$	1	1	1	$\cdots$	1	0	$\bar{Q}^1_1$	$\bar{Q}^1_2$	$\cdots$	$\bar{Q}^1_{\lvert C_1\rvert}$	$\underline{Q}^2_1 + \Delta$	$\underline{Q}^2_2$	$\cdots$	$\underline{Q}^2_{\lvert C_2\rvert}$	0
e_2	1	1	$\cdots$	1	1	1	$\cdots$	1	0	$\bar{Q}^1_1$	$\bar{Q}^1_2$	$\cdots$	$\bar{Q}^1_{\lvert C_1\rvert}$	$\underline{Q}^2_1$	$\underline{Q}^2_2 + \Delta$	$\cdots$	$\underline{Q}^2_{\lvert C_2\rvert}$	0
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
$e_{\lvert C_2\rvert}$	1	1	$\cdots$	1	1	1	$\cdots$	1	0	$\bar{Q}^1_1$	$\bar{Q}^1_2$	$\cdots$	$\bar{Q}^1_{\lvert C_1\rvert}$	$\underline{Q}^2_1$	$\underline{Q}^2_2$	$\cdots$	$\underline{Q}^2_{\lvert C_2\rvert} + \Delta$	0
f	1	1	$\cdots$	1	1	1	$\cdots$	1	1	$\bar{Q}^1_1$	$\bar{Q}^1_2$	$\cdots$	$\bar{Q}^1_{\lvert C_1\rvert}$	$\underline{Q}^2_1$	$\underline{Q}^2_2$	$\cdots$	$\underline{Q}^2_{\lvert C_2\rvert}$	$\bar{Q}_s$

Table 4: Affinely independent points satisfying inequality (42) at equality

C.5 Proof of Theorem 4

Proof: **(Lifting function)** We recall from the lifting procedure described in Section 5.2 that lifting function $F_2(z)$ can be represented by the optimal objective value of the optimization problem

$$F_2(z) = \Gamma + |C| - \min_{y,x} \left\{ \sum_{i\in C} \Big(x_i - (\underline{Q}_i - 1) y_i \Big) \right\} \tag{43a}$$

$$\text{s.t.} \quad \sum_{i \in C} x_i \geq D - z, \tag{43b}$$

$$\underline{Q}_i y_i \leq x_i \leq \bar{Q}_i y_i, \ \forall i \in C, \tag{43c}$$

$$y_i \in \{0, 1\}, \ \forall i \in C. \tag{43d}$$

Note here that without loss of generality, we can assume that $\Gamma \geq 1$, $\underline{Q}_i \geq 1$, and $\bar{Q}_i - \underline{Q}_i \geq 1$ for each $i \in C$. Indeed, the assumption is valid if the data input for Z are integral because Γ, $\underline{Q}_i$ and $\bar{Q}_i - \underline{Q}_i$ for each $i \in C$ are positive, and we can multiply the data input for Z by a sufficiently large positive integer to make them integral. Now we compute $F_2(z)$ by discussing the value of z and solving the corresponding problem (43) in the following cases.

1. If $z \in \big(-\infty, D - \sum_{i\in C} \bar{Q}_i\big)$, we have $\sum_{i\in C} \bar{Q}_i < D - z$. It follows that any point (y, x) satisfying constraints (43c)-(43d) violates constraint (43b) since $\sum_{i\in C} x_i \leq \sum_{i\in C} \bar{Q}_i < D - z$, where the first inequality follows from constraint (43c) and $y_i \leq 1$ for each $i \in C$. Therefore, problem (43) is infeasible and its optimal objective value is ∞. Accordingly, $F_2(z) = -\infty$.

2. If $z \in \big[D - \sum_{i\in C} \bar{Q}_i, A_0\big)$, in view that $A_0 = \Gamma$, we have $D - z > D - \Gamma = \sum_{i\in C} \underline{Q}_i$ by the definition of Γ. We show that the optimal objective value of problem (43) is $\Gamma - z + |C|$. First, we follow a similar argument in the proof of Lemma 3 to show that any given point (y, x) satisfying constraints (43b)-(43d) also satisfies inequality $\sum_{i\in C}\big(x_i - (\underline{Q}_i - 1)y_i\big) \geq \Gamma - z + |C|$, which implies that $\Gamma - z + |C|$ is a lower bound for the optimal objective value of problem (43). To that end, we let $T = \{i \in C : y_i = 0\}$. Accordingly, we have

$$\begin{aligned} D - z \quad &\leq \quad \sum_{i\in C} x_i = \sum_{i\in C} \underline{Q}_i y_i + \sum_{i\in C} (x_i - \underline{Q}_i y_i) \\ &\leq \quad \sum_{i\in C\setminus T} \underline{Q}_i + \sum_{i\in C} (x_i - \underline{Q}_i y_i), \end{aligned}$$

   where the first inequality follows from constraint (43b), and the second inequality follows from the definition of T. It follows from the definition of Γ that $\sum_{i\in C}(x_i - \underline{Q}_i y_i) \geq \Gamma - z + \sum_{i\in T} \underline{Q}_i$.

Therefore, we have

$$\sum_{i\in C}\Big(x_i - (\underline{Q}_i - 1)y_i\Big) = \sum_{i\in C}(x_i - \underline{Q}_i y_i) + \sum_{i\in C} y_i \geq \Gamma - z + \sum_{i\in T}\underline{Q}_i + |C| - |T| \geq \Gamma - z + |C|,$$

where the first inequality is due to the definition of T, and the second inequality follows from the fact that $\underline{Q}_i \geq 1$ for each $i \in T$. Next we show that the lower bound $\Gamma - z + |C|$ can be attained by some feasible solution to problem (43). To that end, we claim that there exists a feasible solution (y^*, x^*) to problem (43) such that $y_i^* = 1$ for each $i \in C$ and $\sum_{i\in C} x_i^* = D - z$. Indeed, $D - z > \sum_{i\in C}\underline{Q}_i$ implies that the solution $y_i^* = 1$ and $x_i^* = \underline{Q}_i$ for each $i \in C$ satisfies constraints (43c)-(43d) but not constraint (43b). To satisfy constraint (43b), we increase the value of x_i^* one by one until $\sum_{i\in C} x_i^* = D - z$, which is well defined because $D - z \leq \sum_{i\in C}\bar{Q}_i$. It follows that $\sum_{i\in C}\big(x_i^* - (\underline{Q}_i - 1)y_i^*\big) = \Gamma - z + |C|$, and so the lower bound is attained. Therefore, the optimal objective value of problem (43) is $\Gamma - z + |C|$ and accordingly $F_2(z) = z$.

3. If $z \in [A_j, A_{j+1} - 1)$ for some $j = 0, \ldots, |C| - 2$, since $A_{|C|} = \Gamma + \sum_{i=1}^{|C|}\underline{Q}_{[i]} = D$ due to the definition of Γ, we have

$$D - z = A_{|C|} - z > A_{|C|} - A_{j+1} + 1 = \sum_{i=j+2}^{|C|}\underline{Q}_{[i]} + 1. \tag{44}$$

We show that the optimal objective value of problem (43) is $|C| - j$. Similar to the proof in the last case, we first show that $|C| - j$ is a lower bound for the optimal objective value by considering any given point (y, x) satisfying constraints (43b)-(43d). We let $T = \{i \in C : y_i = 1\}$ and distinguish two cases on the value of $|T|$. If $|T| \geq |C| - j$, we have

$$\sum_{i\in C}\big(x_i - (\underline{Q}_i - 1)y_i\big) = \sum_{i\in C}\big(x_i - \underline{Q}_i y_i\big) + \sum_{i\in C} y_i \geq \sum_{i\in C} y_i = |T| \geq |C| - j;$$

where the first inequality follows from constraint (43c), and the second equality follows from the definition of T. If $|T| \leq |C| - j - 1$, we have

$$
\begin{aligned}
&\sum_{i\in C}\big(x_i - (\underline{Q}_i - 1)y_i\big) \\
= &\sum_{i\in C} x_i - \sum_{i\in T} \underline{Q}_i + |T| && \text{(by the definition of } T\text{)} \\
\geq &\ D - z - \sum_{i=|C|-|T|+1}^{|C|} \underline{Q}_{[i]} + |T| && \text{(by constraint (43b) and the denition of } \underline{Q}_{[i]}\text{)} \\
> &\ 1 + \sum_{i=j+2}^{|C|-|T|} \underline{Q}_{[i]} + |T| && \text{(by (44))} \\
\geq &\ |C| - j. && \text{(since } \underline{Q}_i \geq 1,\ \forall i \in C\text{)}
\end{aligned}
$$

Next we observe that $|C| - j$ can be attained by a feasible solution (y^*, x^*) to problem (43) with

$$
y^*_{[i]} = \begin{cases} 1, & \text{if } i = j+1, \dots, |C| \\ 0, & \text{o.w.} \end{cases}, \quad x^*_{[i]} = \begin{cases} \underline{Q}_{[i]}, & \text{if } i = j+1, \dots, |C| \\ 0, & \text{o.w.} \end{cases},
$$

where variables $y^*_{[i]}$ and $x^*_{[i]}$ correspond to $\underline{Q}_{[i]}$ for each $i \in C$. (y^*, x^*) is feasible to problem (43) because (i) it clearly satisfies constraints (43c)-(43d), and (ii) it satisfies constraint (43b) since $\sum_{i\in C} x^*_i = \sum_{i=j+1}^{|C|} \underline{Q}_{[i]} = A_{|C|} - A_j \geq D - z$, where the inequality follows from $D = A_{|C|}$ and $z \geq A_j$. Finally, we observe that $\sum_{i\in C}\Big(x^*_i - (\underline{Q}_i - 1)y^*_i\Big) = |C| - j$. Therefore, the optimal objective value of problem (43) is $|C| - j$ and accordingly $F_2(z) = \Gamma + |C| - (|C| - j) = \Gamma + j$.

4. If $z \in [A_{j+1} - 1, A_{j+1})$ for some $j = 0, \dots, |C| - 2$, we have $A_{j+1} - z \leq 1$ and

$$
D - z = A_{|C|} - A_{j+1} + A_{j+1} - z = \sum_{i=j+2}^{|C|} \underline{Q}_{[i]} + A_{j+1} - z, \tag{45}
$$

where the first equality follows from $D = A_{|C|}$. We show that the optimal objective value of problem (43) is $(|C| - j - 1) + (A_{j+1} - z)$. Similar to the proofs in previous cases, we first show that $(|C| - j - 1) + (A_{j+1} - z)$ is a lower bound of the optimal objective value by considering any point (y, x) satisfying constraints (43b)-(43d). To that end, we let $T = \{i \in C : y_i = 1\}$ and distinguish two cases on the value of $|T|$. If $|T| \geq |C| - j$, we have

$$
\begin{aligned}
\sum_{i\in C}\big(x_i - (\underline{Q}_i - 1)y_i\big) &= \sum_{i\in C}(x_i - \underline{Q}_i y_i) + \sum_{i\in C} y_i \ \geq\ \sum_{i\in C} y_i \ =\ |T| \\
&\geq\ |C| - j \ \geq\ (|C| - j - 1) + (A_{j+1} - z),
\end{aligned}
$$

where the first inequality follows from constraint (43c), the second equality follows from the definition of T, and the last inequality is due to $A_{j+1} - z \leq 1$. If $|T| \leq |C| - j - 1$, we have

$$
\begin{aligned}
&\sum_{i\in C}\big(x_i - (\underline{Q}_i - 1)y_i\big) \\
&\geq \sum_{i\in C} x_i - \sum_{i\in T} \underline{Q}_{[i]} + |T| && \text{(by the definition of } T\text{)} \\
&\geq D - z - \sum_{i=|C|-|T|+1}^{|C|} \underline{Q}_{[i]} + |T| && \text{(by constraint (43b) and the definition of } \underline{Q}_i\text{)} \\
&= \sum_{i=j+2}^{|C|} \underline{Q}_{[i]} + A_{j+1} - z - \sum_{i=|C|-|T|+1}^{|C|} \underline{Q}_{[i]} + |T| && \text{(by (45))} \\
&= A_{j+1} - z + \sum_{i=j+2}^{|C|-|T|} \underline{Q}_{[i]} + |T| \\
&\geq |C| - j - 1 + A_{j+1} - z. && \text{(since } \underline{Q}_i \geq 1, \forall i \in C\text{)}
\end{aligned}
$$

Next we observe that $(|C| - j - 1) + (A_{j+1} - z)$ can be attained by a feasible solution (y^*, x^*) to problem (43) with

$$
y^*_{[i]} = \begin{cases} 1, & \forall i = j+2, \ldots, |C| \\ 0, & \text{o.w.} \end{cases}, \quad x^*_{[i]} = \begin{cases} \underline{Q}_{[j+2]} + A_{j+1} - z, & \text{if } i = j+2 \\ \underline{Q}_{[i]}, & \text{if } i = j+3, \ldots, |C| \\ 0, & \text{o.w.} \end{cases},
$$

where variables y^*_i and x^*_i correspond to $\underline{Q}_i$ for each $i \in C$. (y^*, x^*) is feasible to problem (43) because (i) it clearly satisfies constraint (43d), (ii) it satisfies constraint (43c) since $\underline{Q}_{[j+2]} + A_{j+1} - z \leq \underline{Q}_{[j+2]} + 1 \leq \bar{Q}_{[j+2]}$ due to $A_{j+1} - z \leq 1$ and $\bar{Q}_{[j+2]} - \underline{Q}_{[j+2]} \geq 1$ for each $i \in C$, and (iii) it satisfies constraint (43b) since $\underline{Q}_{[j+2]} + A_{j+1} - z + \sum_{i=j+3}^{|C|} \underline{Q}_{[i]} = D - z$ due to (45). Finally, we observe that $\sum_{i\in C}\left(x^*_i - (\underline{Q}_i - 1)y^*_i\right) = (|C| - j - 1) + (A_{j+1} - z)$. Therefore, the optimal objective value of problem (43) is $(|C| - j - 1) + (A_{j+1} - z)$ and accordingly $F_2(z) = \Gamma + |C| - (|C| - j - 1 + A_{j+1} - z) = z - A_{j+1} + 1 + \Gamma + j$.

5. If $z \in [A_{|C|-1}, A_{|C|})$, we have $0 < D - z \leq \underline{Q}_{[|C|]}$ because $A_{|C|} = D$ and $A_{|C|-1} = D - \underline{Q}_{[|C|]}$. We show that the optimal objective value of problem (43) is 1. First, for any given feasible solution (y, x) to problem (43) we have

$$
\sum_{i\in C}\big(x_i - (\underline{Q}_i - 1)y_i\big) = \sum_{i\in C}(x_i - \underline{Q}_i y_i) + \sum_{i\in C} y_i \;\geq\; \sum_{i\in C} y_i \;\geq\; 1,
$$

where the first inequality is due to constraint (43c) and the second inequality follows from the fact that there exists at least one index $i \in C$ with $y_i = 1$ since $D - z > 0$ in constraint (43b). Second, the lower bound 1 is attained by the solution (y^*, x^*) with $y^*_{[|C|]} = 1$, $x^*_{[|C|]} = \underline{Q}_{[|C|]}$ and all their other components zero, where variables $y^*_{[|C|]}$ and $x^*_{[|C|]}$ correspond to $\underline{Q}_{|C|}$. Finally, we observe that (y^*, x^*) is feasible to problem (43) in view that constraint (43b) is satisfied since $\underline{Q}_{|C|} \geq D - z$, and that $\sum_{i \in C} \left(x^*_i - (\underline{Q}_i - 1) y^*_i \right) = 1$. Therefore, the optimal objective value of problem (43) is 1 and accordingly $F_2(z) = \Gamma + |C| - 1$.

6. If $z \geq A_{|C|}$, we have $D - z \leq 0$ since $D = A|C|$. It is clear that the optimal objective value of problem (43) is zero, since it has at least one feasible solution $(0, 0)$. Accordingly, $F_2(z) = \Gamma + |C|$.

(Subadditivity) We prove the subadditivity of the function $\hat{F}_2(z)$ over $\mathbb{R}_+$, whose definition is restated as follows:

$$\hat{F}_2(z) = \begin{cases} -\infty, \quad \text{if } z \in (-\infty, D - \sum_{i \in C} \bar{Q}_i), \\ z, \quad \text{if } z \in [D - \sum_{i \in C} \bar{Q}_i, A_0), \\ \Gamma + j, \quad \text{if } z \in [A_j, A_{j+1} - 1), \ \forall j = 0, \ldots, |C| - 1, \\ z - A_{j+1} + 1 + \Gamma + j, \quad \text{if } z \in [A_{j+1} - 1, A_{j+1}), \ \forall j = 0, \ldots, |C| - 1, \\ \Gamma + |C|, \quad \text{if } z \in [A_{|C|}, \infty). \end{cases} \tag{46}$$

For any given $z_1, z_2 \in \mathbb{R}_+$, we assume without loss of generality that $z_1 \leq z_2$. Since $\hat{F}_2(z)$ is continuous over $\mathbb{R}_+$ and $d\hat{F}_2(z)/dz \leq 1$ for each $z \in \mathbb{R}_+$ where $\hat{F}_2(z)$ is differentiable, we have

$$\frac{\hat{F}_2(z_1 + z_2) - \hat{F}_2(z_2)}{(z_1 + z_2) - z_2} \leq 1 \quad \Leftrightarrow \quad \hat{F}_2(z_1 + z_2) - \hat{F}_2(z_2) \leq z_1. \tag{47}$$

To show that $\hat{F}_2(z_1) + \hat{F}_2(z_2) \geq \hat{F}_2(z_1 + z_2)$, we discuss the following cases based on the values of z_1 and z_2.

1. If $z_1 \in [0, \Gamma - 1)$, in view that $D - \sum_{i \in C} \bar{Q}_i \leq 0$ based on the conditions of Lemma 3 and $A_0 = \Gamma$ by definition, we have $\hat{F}_2(z_1) = z_1$. It follows that $\hat{F}_2(z_1 + z_2) - \hat{F}_2(z_2) \leq z_1 = \hat{F}_2(z_1)$ where the first inequality is due to (47). Therefore, $\hat{F}_2(z_1) + \hat{F}_2(z_2) \geq \hat{F}_2(z_1 + z_2)$.

2. If $z_1 > \Gamma - 1$, we define an auxiliary function $J(z)$ by translating function $\hat{F}_2(z)$ by $\Gamma - 1$ units in both negative z-axis and negative $\hat{F}_2(z)$-axis, i.e., we define

$$J(z) = \hat{F}_2\left(z + (\Gamma - 1)\right) - (\Gamma - 1). \tag{48}$$

For further discussion, we define $A'_j = \sum_{i=1}^{j} \underline{Q}_{[i]}$ for each $j = 0, \ldots, |C|$. Note here that A'_j is obtained by translating A_j by Γ units by the definition of A_j and accordingly $A'_0 = 0$. Under this notation, we have

$$J(z) = \begin{cases} -\infty, \quad \text{if } z \in (-\infty, \sum_{i\in C}(\underline{Q}_i - \bar{Q}_i) + 1), \\ z, \quad \text{if } z \in \left[\sum_{i\in C}(\underline{Q}_i - \bar{Q}_i) + 1, A'_0\right), \\ z - A'_j + j, \quad \text{if } z \in [A'_j, A'_j + 1), \; \forall j = 0, \ldots, |C|, \\ j + 1, \quad \text{if } z \in [A'_j + 1, A'_{j+1}), \; \forall j = 0, \ldots, |C| - 1, \\ |C| + 1, \quad \text{if } z \in [A'_{|C|} + 1, \infty). \end{cases} \tag{49}$$

Now we prove the subadditivity of $\hat{F}_2(z)$ by showing the subadditivity of $J(z)$ over $\mathbb{R}_+$. Assume for now that $J(z)$ is subadditive over $\mathbb{R}_+$, and so we have

$$\begin{aligned} \hat{F}_2(z_1 + z_2) &= \hat{F}_2(z_1 + z_2) - \hat{F}_2(z_1 + z_2 - (\Gamma - 1)) + \hat{F}_2(z_1 + z_2 - (\Gamma - 1)) \\ &\leq \Gamma - 1 + \hat{F}_2(z_1 + z_2 - (\Gamma - 1)) \\ &= \Gamma - 1 + J(z_1 + z_2 - 2(\Gamma - 1)) + (\Gamma - 1) \\ &\leq \Gamma - 1 + J(z_1 - (\Gamma - 1)) + J(z_2 - (\Gamma - 1)) + (\Gamma - 1) \\ &= \hat{F}_2(z_1) + \hat{F}_2(z_2), \end{aligned}$$

where the first inequality is due to (47) and $z_1 + z_2 \geq z_1 + z_2 - (\Gamma - 1) \geq 0$, the second equality is due to (48), the second inequality is due to the subadditivity of $J(z)$ and the fact that $z_2 - (\Gamma - 1) \geq z_1 - (\Gamma - 1) \geq 0$ and $z_1 + z_2 - 2(\Gamma - 1) \geq 0$, and the last equality is due to (48). The remaining task is to show the subadditivity of $J(z)$ which is stated in the following claim.

Claim $J(z)$ is subadditive over $\mathbb{R}_+$.

Proof of claim: For any given $z_1, z_2 \in \mathbb{R}_+$, we assume without loss of generality that $z_1 \leq z_2$. We first extend the definition $A'_j = \sum_{i=1}^{j} \underline{Q}_{[i]}$ for $j = 0, \ldots, |C|$ by defining $\underline{Q}_{[\ell]}$ and A'_ℓ to be ∞ for each integer $\ell \geq |C| + 1$. Under this extended definition, there exists $j, k \in \{0, \ldots, |C|\}$ such that $z_1 \in [A'_j, A'_{j+1})$ and $z_2 \in [A'_k, A'_{k+1})$. Next, to show $J(z_1 + z_2) \leq J(z_1) + J(z_2)$, we observe that there exists $R_1 \in [0, \underline{Q}_{[j+1]})$ and $R_2 \in [0, \underline{Q}_{[k+1]})$ such that $z_1 = A'_j + R_1$ and $z_2 = A'_k + R_2$. By definition of $J(z)$ in (49), we have

$$J(z_1) = j + \min\{R_1, 1\}, \quad J(z_2) = k + \min\{R_2, 1\}, \tag{50}$$

since $J(z_1) = z - A'_j + j = j + R_1$ if $R_1 \in [0, 1)$ and $J(z_1) = j + 1$ if $R_1 \in [1, \underline{\mathrm{Q}}_{[j+1]})$, and the same reasoning holds for $J(z_2)$. Meanwhile, since

$$\begin{aligned} z_1 + z_2 = A'_j + A'_k + R_1 + R_2 &= \sum_{i=1}^{j} \underline{\mathrm{Q}}_{[i]} + \sum_{i=1}^{k} \underline{\mathrm{Q}}_{[i]} + R_1 + R_2 \\ &\leq \sum_{i=1}^{j} \underline{\mathrm{Q}}_{[i]} + \sum_{i=j+1}^{j+k} \underline{\mathrm{Q}}_{[i]} + R_1 + R_2 \\ &= \sum_{i=1}^{j+k} \underline{\mathrm{Q}}_{[i]} + R_1 + R_2 = A'_{j+k} + R_1 + R_2, \end{aligned}$$

where the inequality holds since $\{\underline{\mathrm{Q}}_i\}_{i=1}^{\infty}$ is a nondecreasing order, and the last equality is due to the definition of A'_{j+k}. It follows that

$$J(z_1 + z_2) \leq J(A'_{j+k} + R_1 + R_2),$$

since function $J(z)$ is nondecreasing. Finally, we show $J(z_1 + z_2) \leq J(z_1) + J(z_2)$ by proving $J(A'_{j+k} + R_1 + R_2) \leq J(z_1) + J(z_2)$ and discussing the following cases based on the values of $j + k$ and $R_1 + R_2$.

(a) If $j + k \geq |C| + 1$, then $A'_{j+k} + R_1 + R_2 = \infty$ and accordingly $J(A'_{j+k} + R_1 + R_2) = |C| + 1 \leq j + k \leq J(z_1) + J(z_2)$ by equations (50). Therefore, $J(z_1 + z_2) \leq J(z_1) + J(z_2)$.

(b) If $j+k = |C|$, then $J(A'_{j+k}+R_1+R_2) = j+k+\min\{R_1+R_2, 1\}$ according to the definition of $J(z)$ in (49). Furthermore, we have $\min\{R_1+R_2, 1\} \leq \min\{R_1, 1\}+\min\{R_2, 1\}$. Indeed, if $R_1+R_2 < 1$, then $R_1 < 1$ and $R_2 < 1$ and so $\min\{R_1 + R_2, 1\} = R_1 + R_2 = \min\{R_1, 1\} + \min\{R_2, 1\}$. Otherwise, if $R_1 + R_2 \geq 1$, then $\min\{R_1 + R_2, 1\} = 1$. But in this case $\min\{R_1, 1\} + \min\{R_2, 1\}$ can only take 4 values depending on if R_1 and R_2 are greater than 1, namely $R_1 + 1$, $1 + R_2$, 2, and $R_1 + R_2$, each of which is greater than 1. Hence, we have $\min\{R_1 + R_2, 1\} \leq \min\{R_1, 1\} + \min\{R_2, 1\}$. Therefore, $J(z_1+z_2) \leq j+k+\min\{R_1+R_2, 1\} \leq j+k+\min\{R_1, 1\}+\min\{R_2, 1\} = J(z_1) + J(z_2)$.

(c) If $j + k \leq |C| - 1$, then we distinguish the value of $R_1 + R_2$ in the following cases. Firstly, if $R_1 + R_2 \in [0, \underline{\mathrm{Q}}_{[j+k+1]})$, then $A'_{j+k} + R_1 + R_2 < A'_{j+k+1}$ and accordingly $J(A'_{j+k} + R_1 + R_2) = j+k+\min\{R_1+R_2, 1\}$ based on the definition of $J(z)$ in (49). It follows that $J(z_1+z_2) \leq j+k+\min\{R_1+R_2, 1\} \leq j+k+\min\{R_1, 1\}+\min\{R_2, 1\} = J(z_1)+J(z_2)$ where the second inequality has been proved in part (b). Secondly, if $R_1 + R_2 \geq \underline{Q}_{[j+k+1]}$, then $A'_{j+k} + R_1 + R_2 \geq A'_{j+k+1}$. Meanwhile, $A'_{j+k} + R_1 + R_2 \leq A'_{j+k} + \underline{Q}_{[j+1]} + \underline{Q}_{[k+1]} \leq A'_{j+k+2}$ due to $R_1 \in [0, \underline{Q}_{[j+1]})$ and $R_2 \in [0, \underline{Q}_{[k+1]})$. Hence, $J(A'_{j+k} + R_1 + R_2) = j + k + 1 + \min\{R_1 + R_2 - \underline{Q}_{[j+k+1]}, 1\}$ according to the definition of $J(z)$ in (49). Furthermore, we have $1 + \min\{R_1 + R_2 - \underline{Q}_{[j+k+1]}, 1\} \leq \min\{R_1, 1\} + \min\{R_2, 1\}$. To see that, we discuss the values of R_1 and R_2 in the following subcases.

Subcase 1. If $R_1 \geq 1$ and $R_2 \geq 1$, then $1 + \min\{R_1 + R_2 - \underline{Q}_{[j+k+1]}, 1\} \leq 2 = \min\{R_1, 1\} + \min\{R_2, 1\}$. Therefore, $J(z_1 + z_2) \leq j + k + 1 + \min\{R_1 + R_2 - \underline{Q}_{[j+k+1]}, 1\} \leq j + k + \min\{R_1, 1\} + \min\{R_2, 1\} = J(z_1) + J(z_2)$.

Subcase 2. If only one of R_1 and R_2 is greater than 1, say without loss of generality $R_1 \geq 1$ and $R_2 < 1$, then

$$\begin{aligned} 1 + \min\{R_1 + R_2 - \underline{Q}_{[j+k+1]}, 1\} &\leq 1 + R_1 + R_2 - \underline{Q}_{[j+k+1]} \\ &\leq 1 + R_2 = \min\{R_1, 1\} + \min\{R_2, 1\}, \end{aligned}$$

where the last inequality is due to $R_1 < \underline{Q}_{[j+1]} \leq \underline{Q}_{[j+k+1]}$. Therefore, $J(z_1 + z_2) \leq j + k + 1 + \min\{R_1 + R_2 - \underline{Q}_{[j+k+1]}, 1\} \leq j + k + \min\{R_1, 1\} + \min\{R_2, 1\} = J(z_1) + J(z_2)$.

Subcase 3. If $R_1, R_2 \in [0, 1)$, then

$$\begin{aligned} 1 + \min\{R_1 + R_2 - \underline{Q}_{[j+k+1]}, 1\} &\leq 1 + R_1 + R_2 - \underline{Q}_{[j+k+1]} \\ &\leq R_1 + R_2 = \min\{R_1, 1\} + \min\{R_2, 1\} \end{aligned}$$

where the last inequality is due to $\underline{Q}_{[j+k+1]} \geq 1$. Therefore, $J(z_1 + z_2) \leq j + k + 1 + \min\{R_1 + R_2 - \underline{Q}_{[j+k+1]}, 1\} \leq j + k + \min\{R_1, 1\} + \min\{R_2, 1\} = J(z_1) + J(z_2)$. ■

C.6 Proof of Lemma 6

Proof: First, we prove that $\Phi_i(z) \geq \hat{F}_2(z)$ for each $z \in [\underline{Q}_i, \bar{Q}_i]$. To that end, we claim that $\phi_i^k(z) \geq \hat{F}_2(z)$ for each $z \in [B_i^k, B_i^{k+1}]$ for each $k = 1, \ldots, |\Lambda_i| - 1$. Indeed, by the definitions of B_i^k, B_i^{k+1}, and $\hat{F}_2(z)$ in (46), we have $\hat{F}_2(z) = \max\{\hat{F}_2(B_i^k), z + \hat{F}_2(B_i^{k+1}) - B_i^{k+1}\}$ and

$$0 \leq \frac{\hat{F}_2(B_i^{k+1}) - \hat{F}_2(B_i^k)}{B_i^{k+1} - B_i^k} \leq 1 \tag{51}$$

since $\hat{F}_2(z)$ is continuous and $d\hat{F}_2(z)/dz \le 1$ wherever $\hat{F}_2(z)$ is differentiable. It follows that (i)

$$\phi_i^k(z) = \hat{F}_2(B_i^k) + \left(\frac{\hat{F}_2(B_i^{k+1}) - \hat{F}_2(B_i^k)}{B_i^{k+1} - B_i^k}\right)(z - B_i^k) \ge \hat{F}_2(B_i^k),$$

where the last inequality follows from (51) and $z \ge B_i^k$, and (ii)

$$\begin{aligned}
\phi_i^k(z) &= \hat{F}_2(B_i^k) + \left(\frac{\hat{F}_2(B_i^{k+1}) - \hat{F}_2(B_i^k)}{B_i^{k+1} - B_i^k}\right)(z - B_i^{k+1} + B_i^{k+1} - B_i^k) \\
&= \hat{F}_2(B_i^{k+1}) + \left(\frac{\hat{F}_2(B_i^{k+1}) - \hat{F}_2(B_i^k)}{B_i^{k+1} - B_i^k}\right)(z - B_i^{k+1}) \\
&\ge \hat{F}_2(B_i^{k+1}) + z - B_i^{k+1},
\end{aligned}$$

where the inequality follows from (51) and $z \le B_i^{k+1}$. Hence, $\phi_i^k(z) \ge \hat{F}_2(z)$ on interval $[B_i^k, B_i^{k+1}]$. Furthermore, for two consecutive pieces $\phi_i^k(z)$ and $\phi_i^{k+1}(z)$ for each $k = 1, \ldots, |\Lambda_i| - 2$, we observe that they meet at point $(B_i^{k+1}, \hat{F}_2(B_i^{k+1}))$ and that their slopes satisfy

$$\frac{\hat{F}_2(B_i^{k+2}) - \hat{F}_2(B_i^{k+1})}{B_i^{k+2} - B_i^{k+1}} \le \frac{\hat{F}_2(B_i^{k+1}) - \hat{F}_2(B_i^k)}{B_i^{k+1} - B_i^k}.$$

It follows that $\phi_i^k(z) \le \phi_i^{k+1}(z)$ over $[\underline{Q}_i, B_i^{k+1}]$, and $\phi_i^k(z) \ge \phi_i^{k+1}(z)$ over $[B_i^{k+1}, \bar{Q}_i]$. Therefore, by following similar arguments for each $k = 1, \ldots, |\Lambda_i| - 2$ we have $\Phi_i(z) = \phi_i^k(z)$ over each interval $[B_i^k, B_i^{k+1}]$, and thus $\Phi_i(z) \ge \hat{F}_2(z)$ on interval $[\underline{Q}_i, \bar{Q}_i]$ since $\phi_i^k(z) \ge \hat{F}_2(z)$.

Second, $\phi_i^k(z)$ touches $\hat{F}_2(z)$ at two points $(B_i^k, \hat{F}_2(B_i^k))$ and $(B_i^{k+1}, \hat{F}_2(B_i^{k+1}))$ by the definition of $\phi_i^k(z)$. ■

C.7 Proof of Proposition 10

Proof: First, since $\bar{Q}_i \le D - 1$ for all $i \in \mathcal{I}$, we have $F_2(z) = \hat{F}_2(z)$ for all $z \in [\underline{Q}_i, \bar{Q}_i]$ for each $i \in \mathcal{I}$. Hence, $F_2(z)$ is subadditive on interval $[\underline{Q}_i, \bar{Q}_i]$ for each $i \in \mathcal{I} \setminus C$ and it can be used to lift pair (y_i, x_i) for each $i \in \mathcal{I} \setminus C$ in a sequence independent manner.

Second, to lift pair (y_i, x_i) for each $i \in \mathcal{I} \setminus C$, since the function $\Phi_i(z)$ defined in (31) is a piecewise linear overestimation of $F_2(z)$, we can choose (α_i, β_i) as the intersect and slope of the linear piece $\phi_i^k(z)$ to obtain a valid lifting coefficient for any $k = 1, \ldots, |\Lambda_i| - 1$. Since $\phi_i^k(z) = F_2(B_i^k) + \left(\frac{F_2(B_i^{k+1}) - F_2(B_i^k)}{B_i^{k+1} - B_i^k}\right)(z - B_i^k)$ by Lemma 6, we choose

$$\alpha_i = \frac{F_2(B_i^k)B_i^{k+1} - F_2(B_i^{k+1})B_i^k}{B_i^{k+1} - B_i^k}, \quad \beta_i = \frac{F_2(B_i^{k+1}) - F_2(B_i^k)}{B_i^{k+1} - B_i^k},$$

and so the lifted inequality (32) is valid for $\mathrm{conv}(Z)$.

Third, to prove the dimension of the face defined by the lifted inequality (32), we claim that its seed inequality (20) defines a face of $\mathrm{conv}(Z_C)$ of dimension at least $|C|-1$. To this end, we generate $|C|$ affinely independent points $(\bar{y}^j, \bar{x}^j) \in \mathrm{conv}(Z_C)$ for $j = 1, \ldots, |C|$ which satisfy inequality (20) at equality. Note here that since $D < \sum_{i\in C} \bar{Q}_i$ due to a condition of Lemma 3, there exists a set of generation quantities $\{x_i^* : i \in C\}$ such that $\underline{Q}_i < x_i^* < \bar{Q}_i$ and $\sum_{i\in C} x_i^* = D + \epsilon$ for a sufficiently small positive number ϵ with $\epsilon \leq \sum_{i\in C} \bar{Q}_i - D$ and $\epsilon \leq x_i^* - \underline{Q}_i$ for each $i \in C$. (Intuitively, since the total generation capacity $\sum_{i\in C} \bar{Q}_i$ is strictly greater than load D, there exists a set of generation quantity that can over-satisfy D by a small amount.) It follows that $(\bar{y}^j, \bar{x}^j) \in \mathrm{conv}(Z_C)$ for each $j = 1, \ldots, |C|$ with

$$\bar{y}_i^j = 1, \quad \bar{x}_i^j = \begin{cases} x_i^* - \epsilon, & \text{if } i = j, \\ x_i^*, & \text{o.w.} \end{cases}, \ \forall i \in C,$$

since $\sum_{i\in C}(\bar{x}_i^j - (\underline{Q}_i - 1)\bar{y}_i^j) = \sum_{i\in C} \bar{x}_i^* - \epsilon - \sum_{i\in C} \underline{Q}_i + |C| = D - \sum_{i\in C} \underline{Q}_i + |C| = \Gamma + |C|$, where the last equality follows from the definition of Γ. Moreover, it is clear that points $\{(\bar{y}^j, \bar{x}^j) : j = 1, \ldots, |C|\}$ are affinely independent and accordingly seed inequality (20) defines a face of $\mathrm{conv}(Z_C)$ of dimension at least $|C| - 1$.

Furthermore, for each $k = 1, \ldots, |\Lambda_i| - 1$, $\phi_i^k(z)$ touches $F_2(z)$ at two points $\big(B_i^k, F_2(B_i^k)\big)$ and $\Big(B_i^{k+1}, F_2(B_i^{k+1})\Big)$, and these two points are linearly independent for all $i \in \mathcal{I} \setminus C$ with exception that $\bar{Q}_i \geq \Gamma$, in which case these two points are parallel. Hence, lifting pair (y_i, x_i) provides two linear independent points for each $i \in \mathcal{I} \setminus (C \cup T)$, and one point for each $i \in T$. Therefore, after lifting all the pairs (y_i, x_i) for $i \in \mathcal{I} \setminus C$, the lifted inequality defines a face of $\mathrm{conv}(Z)$ of dimension at least $|C| - 1 + 2(I - |C| - |T|) + |T| = 2I - |C| - |T| - 1$, where $|C| - 1$ is the dimension of the seed inequality, $2(I - |C| - |T|)$ is provided by lifting (y_i, x_i) for $i \in \mathcal{I} \setminus (C \cup T)$, and $|T|$ is provided by lifting (y_i, x_i) for $i \in T$. ∎

C.8 Proof of Proposition 11

Proof: We recall from the lifting procedure described in Section 5.2 that lifting function $F_3(z)$ can be represented by the optimal objective value of the optimization problem

$$
\begin{aligned}
F_3(z) = |C| - 1 - \min_{y,x} \quad & \Big\{\sum_{i\in C} y_i\Big\} && \text{(52a)}\\
\text{s.t.} \quad & \sum_{i\in C} x_i \ge D - z, && \text{(52b)}\\
& \underline{Q}_i y_i \le x_i \le \bar{Q}_i y_i,\ \forall i \in C, && \text{(52c)}\\
& y_i \in \{0,1\},\ \forall i \in C, && \text{(52d)}
\end{aligned}
$$

Now we compute $F_3(z)$ by discussing the value of z and solving the corresponding problem (52) in the following cases.

1. If $z \in (-\infty, D - \sum_{i\in C} \bar{Q}_i)$, we have $\sum_{i\in C} \bar{Q}_i < D - z$. It follows that any point (y, x) satisfying constraints (52c)-(52d) violates constraint (52b) since $\sum_{i\in C} x_i \le \sum_{i\in C} \bar{Q}_i < D - z$, where the first inequality follows from constraint (52c) and $y_i \le 1$ for each $i \in C$. Therefore, problem (52) is infeasible and its optimal objective value is ∞. Accordingly, $F_3(z) = -\infty$.

2. If $z \in [D - \sum_{i\in C} \bar{Q}_i, D - \sum_{i=2}^{|C|} \bar{Q}_{[i]})$, we have $\sum_{i=2}^{|C|} \bar{Q}_{[i]} < D - z \le \sum_{i\in C} \bar{Q}_i$ and so $\sum_{i=2}^{|C|} \bar{Q}_{[i]} < \sum_{i\in C} x_i$ by constraint (52b). We show that the optimal objective value of problem (52) is $|C|$. To that end, we consider any feasible solution (y, x) to problem (52) (note here that such feasible solution exists because $D - z \le \sum_{i\in C} \bar{Q}_i$ and we can set $y_i = 1$ and $x_i = \bar{Q}_i$ to satisfy constraints (52b)-(52d)). We claim that $y_i = 1$ for each $i \in C$ and prove this claim by contradiction. Assume that $\sum_{i\in C} y_i \le |C| - 1$, then we have

   $$\sum_{i\in C} x_i \ \le\ \sum_{i\in C} \bar{Q}_i y_i \ \le\ \sum_{i=2}^{|C|} \bar{Q}_{[i]},$$

   where the first inequality follows from constraint (52c), and the second inequality is due to $\sum_{i\in C} y_i \le |C| - 1$ and the definition of $\bar{Q}_{[i]}$, which contradicts with the fact that $\sum_{i=2}^{|C|} \bar{Q}_{[i]} < \sum_{i\in C} x_i$. Hence, $y_i = 1$ for each $i \in C$ and so $\sum_{i\in C} y_i = |C|$, which implies that $|C|$ is a lower bound of the optimal objective value of problem (52). But $\sum_{i\in C} y_i \le |C|$ due to constraint (52d), and so $|C|$ is also an upper bound. Therefore, the optimal objective value of problem (52) is $|C|$ and accordingly $F_3(z) = -1$.

3. If $z \in [D - \sum_{i=j+1}^{|C|} \bar{Q}_{[i]}, D - \sum_{i=j+2}^{|C|} \bar{Q}_{[i]})$ for $j = 1, \ldots, |C| - 1$, i.e., if $z \in [D - \sum_{i=2}^{|C|} \bar{Q}_{[i]}, \Omega)$ (for $j = 1$) or if $z \in [\Omega + G_j, \Omega + G_{j+1})$ for $j = 2, \ldots, |C| - 1$ by the definitions of Ω and G_j, then we have $\sum_{i=j+2}^{|C|} \bar{Q}_{[i]} < D - z \leq \sum_{i=j+1}^{|C|} \bar{Q}_{[i]}$. It follows that $\sum_{i=j+2}^{|C|} \bar{Q}_{[i]} < \sum_{i\in C} x_i$ due to constraint (52b). We show that the optimal objective value of problem (52) is $|C| - j$. To that end, we first claim that any feasible solution (y, x) to problem (52) satisfies $\sum_{i\in C} y_i \geq |C| - j$, which implies that $|C| - j$ is a lower bound of the optimal objective value of problem (52). We prove by contradiction and assume that $\sum_{i\in C} y_i \leq |C| - j - 1$. It follows that

$$\sum_{i\in C} x_i \;\leq\; \sum_{i\in C} \bar{Q}_i y_i \;\leq\; \sum_{i=j+2}^{|C|} \bar{Q}_{[i]},$$

where the first inequality follows from constraint (52c), and the second inequality is due to $\sum_{i\in C} y_i \leq |C|-j-1$ and the definition of $\bar{Q}_{[i]}$, which contradicts with the fact that $\sum_{i=j+2}^{|C|} \bar{Q}_{[i]} < \sum_{i\in C} x_i$. Hence, $|C| - j$ is a lower bound of the optimal objective value of problem (52). Next we claim that this lower bound can be attained by a feasible solution (y^*, x^*) with

$$y^*_{[i]} = \begin{cases} 0, & \text{if } i = 1, \ldots, j, \\ 1, & \text{if } i = j+1, \ldots, |C|, \end{cases} \qquad x^*_{[i]} = \begin{cases} 0, & \text{if } i = 1, \ldots, j, \\ \bar{Q}_{[i]}, & \text{if } i = j+1, \ldots, |C|, \end{cases}$$

where $y^*_{[i]}$ and $x^*_{[i]}$ correspond to $\bar{Q}_{[i]}$ for each $i \in C$. (y^*, x^*) is feasible because (i) it clearly satisfies constraints (52c)-(52d), and (ii) it satisfies constraint (52b) due to $D - z \leq \sum_{i=j+1}^{|C|} \bar{Q}_{[i]}$. Finally, $\sum_{i\in C} y^*_i = |C| - j$ and so the optimal objective value of problem (52) is $|C| - j$, and accordingly $F_3(z) = j - 1$.

4. If $z \geq \Omega + G_{|C|}$, then we have $0 \geq D - z$ by the definitions of Ω and $G_{|C|}$. It follows that $(y^*, x^*) = (0, 0)$ is a feasible solution to problem (52) since it satisfies constraints (52b)-(52d), and so zero is an upper bound of the optimal objective value of problem (52). On the other hand, zero is also a lower bound because any feasible solution (y, x) to problem (52) satisfies $\sum_{i\in C} y_i \geq 0$. Therefore, the optimal objective value of problem (52) is zero and accordingly $F_3(z) = |C| - 1$. ■

C.9 Proof of Lemma 7

Proof: By following a similar reasoning to the one given in Lemma 6, we can show that $\Psi_i(z)$ is an overestimation of $\hat{F}_3(z)$ on $[\underline{Q}_i, \bar{Q}_i]$ for each $i \in \mathcal{I} \setminus C$, and each $\psi_i^k(z)$ touches $\hat{F}_3(z)$ at two points $(H_i^k, \hat{F}_3(H_i^k))$ and $(H_i^{k+1}, \hat{F}_3(H_i^{k+1}))$ for each $k = 1, \dots, |\Pi_i| - 1$. Here we omit the proof due to similarity. We prove the tightness of the overestimation, i.e. each $\psi_i^k(z)$ touches $F_3(z)$ at two points with possible exception when $k = 1$ or $|\Pi_i| - 1$. By the definitions of $F_3(z)$ and $\hat{F}_3(z)$ (see Figure 4 for comparison), their function values are different on intervals $(G_{j+1}, \Omega + G_{j+1})$ for $j = 1, \dots, |C| - 1$, and are the same elsewhere on $\mathbb{R}_+$. Meanwhile, the definition of Π_i implies that $H_i^1 = \underline{Q}_i$, $H_i^{|\Pi_i|} = \bar{Q}_i$, and $H_i^k = \Omega + G_\ell$ for some $\ell = 2, \dots, |C|$ for each $k = 2, \dots, |\Pi_i| - 1$. Hence, when $k = 1$, it is possible that $\underline{Q}_i$ is within one of the intervals $(G_{j+1}, \Omega + G_{j+1})$ for $j = 1, \dots, |C| - 1$, such that the piece $\psi_i^k(z)$ does not touch $F_3(z)$ at point $(H_i^k, F_3(H_i^k))$ since $F_3(H_i^k) \neq \hat{F}_3(H_i^k)$. Similarly, when $k = |\Pi_i| - 1$, it is possible that $\bar{Q}_i$ is within one of the intervals $(G_{j+1}, \Omega + G_{j+1})$ for $j = 1, \dots, |C| - 1$, such that the piece $\psi_i^k(z)$ does not touch $F_3(z)$ at point $(H_i^{k+1}, F_3(H_i^{k+1}))$. On the other hand, when $k = 2, \dots, |\Pi_i| - 2$, $\psi_i^k(z)$ touches $\hat{F}_3(z)$ and hence $F_3(z)$ at two points $(H_i^k, F_3(H_i^k))$ and $(H_i^{k+1}, F_3(H_i^{k+1}))$ since $F_3(H_i^k) = \hat{F}_3(H_i^k)$ and $F_3(H_i^{k+1}) = \hat{F}_3(H_i^{k+1})$. ■

C.10 Proof of Proposition 13

Proof: The proof for validity is similar to the one given in Proposition 10 and omitted. We prove that the lifted inequality (35) is facet-defining for $\text{conv}(Z)$ under the conditions $|\Pi_i| \geq 4$ and $k = 2, \dots, |\Pi_i| - 2$ for each $i \in \mathcal{I} \setminus C$. First, the seed inequality $\sum_{i \in C} y_i \geq |C| - 1$ is facet-defining for $\text{conv}(Z_C)$ based on Lemma 4. Second, since $|\Pi_i| \geq 4$ and $k = 2, \dots, |\Pi_i| - 2$ for each $i \in \mathcal{I} \setminus C$, we obtain lifting coefficients (α_i, β_i) in the lifted inequality (35) by using the tight lifting pieces $\phi_i^k(z)$ for each $i \in \mathcal{I} \setminus C$. That is, each piece $\phi_i^k(z)$ touches the original lifting function $F_3(z)$ at two points $(H_i^k, F_3(H_i^k))$ and $(H_i^{k+1}, F_3(H_i^{k+1}))$ by Lemma 7. Furthermore, by definition of $F_3(z)$, $(F_3(H_i^{k+1}) - F_3(H_i^k))/(H_i^{k+1} - H_i^k) < 1$ when $k = 2, \dots, |\Pi_i| - 2$, and thus these two points are linearly independent. The final conclusion follows from Lemma 5. ■